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Homework on Bayesian Games

Sketch of the Suggested Solutions

Exercise 1: (Final 2020, 15 points) Consider the Cournot duopoly model in which two firms, 1 and 2, simultaneously choose the quantities they supply, q_1 and q_2. The price each will face is determined by the market demand function $p(q_1, q_2) = a - b(q_1 + q_2)$. Each firm has a probability μ of having a marginal unit cost of c_L and a probability $1-\mu$ of having marginal unit cost of c_H. These probabilities are common knowledge, but the true type is revealed only to each firm individually. Solve for the *Bayesian Nash equilibrium*.

For player i with cost (type) c_i, the payoff as a function of chosen output levels:

$$v_i = (a - bq_i - bEq_j - c_i)\, q_i$$

where Eq_j is the expected output level from the other player j, that is $Eq_j = \mu q_{jL} + (1-\mu)\, q_{jH}$.

Player i chooses q_i to maximize v_i. The first order condition for this maximization is:

$$\begin{aligned} a - bq_i - bEq_j - c_i - bq_i &= 0 \\ a - 2bq_i - bEq_j - c_i &= 0 \end{aligned}$$

giving the best response function of player i with type c_i as

$$q_i = \frac{a - c_i}{2b} - \frac{1}{2}Eq_j$$

Since negative output levels are ruled out, $q_i = \max\left\{0, \frac{a-c_i}{2b} - \frac{1}{2}Eq_j\right\}$. But I will ignore this issue and assume that $\frac{a-c_i}{2b} - \frac{1}{2}Eq_j$ is positive in equilibrium (c_L and c_H are both low enough).

A Bayesian Nash Equilibrium should give us the output levels of the two types of the two agents: $q_{1L}, q_{1H}, q_{2L}, q_{2H}$. And each of these should be a best response to the output levels chosen by the other agent. This yields four equations to solve for the four unknowns.

An easy way to solve this system of equations is to take the expectation over the best response function of agent with type c_i that we found above:

$$Eq_i = \frac{a - Ec}{2b} - \frac{1}{2}Eq_j$$

where Ec is the expected cost, that is $Ec = \mu c_L + (1-\mu)\, c_H$. Using the symmetry of the players, we can see that $Eq_i = Eq_j$ and therefore

$$\begin{aligned} \frac{3}{2}Eq_i &= \frac{a - Ec}{2b} \\ Eq_i &= Eq_j = \frac{a - Ec}{3b} \end{aligned}$$

Substituting the value of Eq_j in the best-response function, we find

$$\begin{aligned} q_i &= \frac{a-c_i}{2b} - \frac{a-Ec}{6b} \\ &= \frac{2a-3c_i+Ec}{6b} \\ &= \frac{2a-3c_i+\mu c_L+(1-\mu)\,c_H}{6b} \end{aligned}$$

implying that

$$\begin{aligned} q_{1L} &= q_{2L} = \frac{2a-(3-\mu)\,c_L+(1-\mu)\,c_H}{6b} \\ q_{1H} &= q_{2H} = \frac{2a-(2+\mu)\,c_H+\mu c_L}{6b} \end{aligned}$$

Notice that, when the values of c_L and c_H are low enough, the output levels above are indeed non-negative and we have a valid Bayesian Nash equilibrium.

Exercise 2: (Final 2019, 30 points) *Bayesian Games.* Two friends – we will call them Anatoli and Milad – will *simultaneously* decide which movie to go (they don't have much of a communication). The alternatives are The Shining and The Blues Brothers. There is a possibility that Anatoli had a nightmare the previous night. Anatoli *knows* whether he had a nightmare or not. Milad *does not know* whether Anatoli had a nightmare the previous night, but he thinks both possibilities are equally likely (probability 1/2).

The payoffs in case Anatoli had a peaceful sleep are (Anatoli is the "row-player", Milad is the "column-player") given in the following table:

		Milad	
		Shining	Blues Brothers
Anatoli	Shining	$2,1$	$0,0$
	Blues Brothers	$0,0$	$1,2$

All of this is common knowledge between the two players.

(5) a) How many pure-strategy Bayesian strategies does each player have?

Milad has two strategies: S and B. Since Anatoli has two types and a Bayesian strategy is a function from types into the available actions, Anatoli has four Bayesian strategies: SS, SB, BS, and BB.

(10) b) Find all pure-strategy Bayesian-Nash equilibria if $x = 3$.

The tables look like this:

		Milad	
		Shining	Blues Brothers
Anatoli	Shining	$2,1$	$0,0$
	Blues Brothers	$0,0$	$1,2$

		Milad	
		Shining	Blues Brothers
Anatoli	Shining	$-1, 1$	$-3, 0$
	Blues Brothers	$0, 0$	$1, 2$

We start with the one-type player Milad.

Suppose that Milad plays S for sure. Then Anatoli of type a chooses S instead of B (2>0) and Anatoli of type b chooses B instead of S (0>-1). S is indeed the BR for Milad if the corresponding payoff is at-least as big as the payoff from B:

$$\begin{aligned} 1p + 0(1-p) &\geq 0p + 2(1-p) \\ p &\geq 2 - 2p \\ p &\geq \frac{2}{3} \end{aligned}$$

Hence, when $p \geq \frac{2}{3}$, the strategy combination $\{S, SB\}$ is a BNE. Since we know that $p = \frac{1}{2}$, $\{S, SB\}$ is not a BNE.

Suppose now that Milad plays B for sure. Then Anatoli of type a chooses B (1>0) and Anatoli of type b chooses also B (1>-3). B is indeed the BR for Milad if the corresponding payoff is at-least as big as the payoff from S:

$$\begin{aligned} 2p + 2(1-p) &\geq 0p + 0(1-p) \\ 2p &\geq 0 \end{aligned}$$

Hence, for any p (so also for $p = \frac{1}{2}$), the strategy combination $\{B, BB\}$ is a BNE.

(15) c) Find all pure-strategy Bayesian-Nash equilibria if $x = 1$.

We did this in class.