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Sketch of the Suggested Solutions for Homework on Mixed Strategies

Exercise 1: (Final 2018) Find all the Nash equilibria (in pure and/or mixed strategies) of the following game

		Player 2	
		Left	*Right*
Player 1	*Top*	6, 2	0, 1
	Bottom	3, 0	6, 4

Solution 1: $(Top, Left)$ and $(Bottom, Right)$ are the two pure-strategy Nash equilibria. To find the other equilibria, let p be the probability that player 1 will choose top and q be the probability that player 2 will choose left. Find out the best response functions: Player 1 (strictly) prefers Top to $Bottom$ if $q > 2/3$ and Player 2 (strictly) prefers $Left$ to $Right$ if $p > 4/5$. Accordingly, in addition to the pure-strategy Nash equilibria $(p = 1, q = 1)$ and $(p = 0, q = 0)$, there is also a mixed-strategy Nash equilibrium where $(p = 4/5, q = 2/3)$.

(20) **Exercise 2:** (Final 2017) Consider the following game between the taxpayer and an auditor. The taxpayer can save x euros by evading taxes. But if a tax-evader is audited, he has to pay a fine f which is higher than the taxes saved x. The auditor incurs cost c when she decides to audit. But if she catches an evasion, she receives a reward r which is higher than the auditing cost c. Assuming that each player is maximizing the expected monetary returns (ignoring ethical considerations), the payoff matrix can be written as follows:

	Audit	*No Audit*
Evade	$x - f, r - c$	$x, 0$
Not Evade	$0, -c$	$0, 0$

Find the Nash equilibria of this game. Government officials want to reduce the rate of tax evasion. What is your recommendation? Should they change the fine level? Should they change the reward paid to the auditor? What should be the direction of the change(s)? Explain briefly.

Solution: There is no pure-strategy Nash equilibrium: The taxpayer wants to evade taxes if he knows that the auditor is not auditing. But the auditor would want to audit if she thinks that the taxpayer is evading. To find the mixed-strategy Nash equilibria, let p be the probability that the taxpayer evades and q be the probability that the auditor audits.

For the taxpayer, the expected payoff to evading tax is $x - qf$, and payoff to not evading is zero. For the auditor, expected payoff to auditing is $pr - c$, and payoff to not auditing is zero. Accordingly, the best response correspondences are

$$p = \begin{cases} 0 & \text{if } q > x/f \\ [0,1] & \text{if } q = x/f \\ 1 & \text{if } q < x/f \end{cases}$$

for the taxpayer and

$$q = \begin{cases} 1 & \text{if } p > c/r \\ [0,1] & \text{if } p = c/r \\ 0 & \text{if } q < c/r \end{cases}$$

for the auditor. The unique Nash equilibrium of the game is $p = c/r$ and $q = x/f$. The rate of tax evasion (p) does not change if we increase the fine level f. But it is decreasing in the reward r that is paid to the successful auditor.

(25) **Exercise 3:** (Midterm 2017) Consider the following game where the pure strategies for Player 1 are U and D, and the pure strategies for Player 2 are L and R:

	L	R
U	$3,0$	$1,0$
D	$1,2$	$3,1$

(5) a) Is there a *strictly dominated* pure strategy for any of the players? If yes, which one?

No strictly dominated strategy for the players.

(5) b) Is there a *weakly dominated* pure strategy for any of the players? If yes, which one?

R is weakly dominated by L for player 2.

(5) c) Are there Nash equilibria of this game in *pure strategies*? If yes, find them.

Through circling the payoffs, you can see that (U,L) is a Nash equilibrium in pure strategies.

(10) d) Find *all* the Nash equilibria of this game.

The safest way to find all the Nash equilibria in mixed strategies is drawing the best response correspondences. Let's say p is the probability that player 1 plays U and q is the probability that player 2 plays L. Best response for Player 1 is similar to the best response function in the matching pennies or the battle of the sexes games:

$$p = \begin{cases} 0 & \text{if } q < 1/2 \\ [0,1] & \text{if } q = 1/2 \\ 1 & \text{if } q > 1/2 \end{cases}$$

Best response for Player 2 is rather non-standard. Notice that playing the weakly dominant strategy of L is always a part of the best response. When Player 1 plays D with some positive probability (when p<1), L is the only best response. But when Player 2 is sure that player 1 is playing U (p=1), she is indifferent between L and R. Accordingly, her best response is

$$q = \begin{cases} 1 & \text{if } p < 1 \\ [0,1] & \text{if } p = 1 \end{cases}$$

Drawing these two correspondences on the $p \times q$ coordinate system identifies a continuum of mixed strategy Nash equilibria: $\{(p,q) : p = 1 \text{ and } 1/2 \leq q \leq 1\}$. That is, in the Nash equilibria, Player 1 plays U for sure and Player 2 plays L with some probability q, where q can be any number between 1/2 and 1 (including 1/2 and 1). Notice that the pure-strategy Nash equilibrium (U,L) is one of the identified Nash equilibria ($p = 1, q = 1$).

(25 points) **Exercise 4:** (Midterm 2018) Consider the game with the following payoff matrix, played by players 1 and 2:

Player 2

Player 1		*Left*	*Right*
	Top	x, y	$0, 0$
	Bottom	$0, 0$	$1, 1$

where x and y are **positive numbers**. As usual, the first payoff in each cell belongs to player 1 - who chooses between the rows and the second payoff belongs to player 2 - who chooses between the columns.

(5) a) Find the *pure strategy* Nash equilibria of this game.

(Top, Left) and (Bottom, Right) are the two pure-strategy Nash Eq.

For the rest of the exercise, consider the mixed strategies.

(10) b) Write down the best response correspondence of each player in this game.

Let p be the probability that player 1 chooses Top and q be the probability that player 2 chooses Left. We can refer to p and q as the mixed strategies of these two players. The best response correspondence of player 1 should give the expected payoff maximizing level(s) of p for him for each level of q.

$$\begin{aligned} Ev_1(Top) &= qx + (1-q)\,0 = qx \\ Ev_1(Bottom) &= q0 + (1-q)\,1 = 1-q \\ qx &> 1-q \Leftrightarrow (1+x)\,q > 1 \Leftrightarrow q > 1/(1+x) \end{aligned}$$

Player 1 strictly prefers playing Top when $q > 1/(1+x)$, and strictly prefers Bottom when $q < 1/(1+x)$, and he is indifferent between the two options (and any randomization between them) when $q = 1/(1+x)$. For player 1, recall that playing Top for sure means setting p = 1 and playing Bottom for sure is p = 0. Accordingly, the best response correspondence of player 1 is

$$BR_1(q) = p^*(q) = \begin{cases} 0 & q < 1/(1+x) \\ [0,1] & q = 1/(1+x) \\ 1 & q > 1/(1+x) \end{cases}$$

From symmetry between the payoffs of the players, the best response correspondence for player 2 is

$$BR_2(p) = q^*(p) = \begin{cases} 0 & p < 1/(1+y) \\ [0,1] & p = 1/(1+y) \\ 1 & p > 1/(1+y) \end{cases}$$

(5) c) Find all the Nash equilibria of this game.

A Nash equilibrium in mixed strategies is defined as (p, q) such that $p = BR_1(q)$and $q = BR_2(p)$. In addition to the two pure-strategy Nash equilibria in part (b), there is one Nash equilibrium in completely mixed strategies:

Player 1 chooses Top with probability 1/(1 + y) and Bottom with probability y/(1 + y).

Player 2 chooses Left with probability 1/(1 + x) and Right with probability x/(1 + x).

To persuade yourself further that these strategies constitute a Nash equilibrium:

If Player 1 randomizes with these probabilities, Player 2 receives expected payoff $\frac{1}{1+y}y$ from playing left and $\frac{y}{1+y}1$ from playing right. Since these payoffs are the same, Player 2 is indifferent. It is optimal for her to play any randomization between the two pure strategies.

If Player 2 randomizes with these probabilities, Player 1 receives expected payoff $\frac{1}{1+x}x$ from playing top and $\frac{x}{1+x}1$ from playing bottom. Since these payoffs are the same, Player 1 is indifferent. It is optimal for her to play any randomization between the two pure strategies.

(5) d) Suppose that the players are playing a Nash equilibrium of this game in *completely* mixed strategies (not in pure strategies). How would increasing the payoff x change the probability that player 1 choose to play Top? Explain in one or two sentences.

Increasing the payoff of player 1 here (x) would not change his equilibrium strategy in a mixed-strategy Nash equilibrium. The equilibrium probabilities of different alternatives for player 1 are chosen in order to keep player 2 indifferent between her alternatives.

Exercise 5: (Final 2018) Consider the following game between two players. Each of the players will simultaneously choose a number. The player who chooses a (strictly) higher number wins and the other one loses. If they choose the same number, there is a tie between the players. Each player prefers to win rather than to tie; and he prefers to tie, rather than to lose. Is there a Nash equilibrium of this game? Reconcile your answer with the existence theorem of Nash.

Solution 5: There is no Nash equilibrium in this game. Whatever number (or probability distribution over some numbers) a player chooses, the other one can win the game by just choosing a higher number. So whatever pair of strategies you pick for the player, there is a profitable deviation for at least one of them. This is not a contradiction to Nash's existence theorem, since the theorem is about *finite* games, where each player has finitely many pure strategies.

Exercise 6: (Midterm 2019, 20 points) Bad news. Alex and Chris got separated. It was a painful breakup and they do not talk to each other anymore. But life goes on. Now each of them has to decide on which classes to register for the Spring term. There are two options. Option A is a course on *Advanced Game Theory*. Option B is *Basics of the Astrological Dimensions of Business*. Since it will be painful to see each other in every lecture, Alex and Chris will both get zero payoff if they register to the same class. If they choose to register to different classes, the payoffs are positive, but the one who chooses the advanced game theory course will get x times as much payoff as the other.

(10) a) Write down the normal form (with a payoff matrix) and find all Nash equilibria (in pure and mixed strategies) of this game.

		Chris	
		A	B
Alex	A	$0,0$	$x,1$
	B	$1,x$	$0,0$

pure strategy Nash equilibria: (A,B) and (B,A). To find all the Nash equilibria, let p be the probability that player 1 chooses A and q be the probability that player 2 chooses A. We can refer to p and q as the mixed strategies of these two players. The best response correspondence of player 1 should give the expected payoff maximizing level(s) of p for him for each level of q.

$$\begin{aligned} Ev_1(A) &= q0 + (1-q)x = (1-q)x \\ Ev_1(B) &= q1 + (1-q)0 = q \\ (1-q)x &> q \Leftrightarrow x > (1+x)q \Leftrightarrow q < x/(1+x) \end{aligned}$$

Player 1 strictly prefers playing A when $q < x/(1+x)$, and strictly prefers B when $q > x/(1+x)$, and he is indifferent between the two options (and any randomization between them) when $q = x/(1+x)$. For player 1, recall that playing A for sure means setting $p = 1$ and playing B for sure is $p = 0$. Accordingly, the best response correspondence of player 1 is

$$b_1(q) = p^*(q) = \begin{cases} 1 & q < x/(1+x) \\ [0,1] & q = x/(1+x) \\ 0 & q > x/(1+x) \end{cases}$$

From symmetry between the payoffs of the players, the best response correspondence for player 2 is

$$b_2(p) = q^*(p) = \begin{cases} 1 & p < x/(1+x) \\ [0,1] & p = x/(1+x) \\ 0 & p > x/(1+x) \end{cases}$$

A Nash equilibrium in mixed strategies is defined as (p, q) such that $p = b_1(q)$and $q = b_2(p)$. In addition to the two pure-strategy Nash equilibria in part (b), there is one Nash equilibrium in completely mixed strategies:

$$p = q = \frac{x}{1+x}$$

(10) b) You should notice that one of the equilibria you found above is symmetric, i.e. both players follow the same strategy. Find out if this symmetric equilibrium strategy is evolutionary stable: Assume that this strategy describes the proportions of players in a large population that are programmed to make different choices each time they are randomly matched with another player. Suppose the payoffs give the relative survival rates of the players. If the equilibrium proportions are modified, would the evolutionary forces push them back to the initial state in this game? Why or why not?

Yes, $p = \frac{x}{1+x}$ is an evolutionary stable strategy. If the proportion of players p choosing strategy A is higher than $\frac{x}{1+x}$, then players choosing option B would make higher profits:

$$v(A) = p0 + (1-p)x < \frac{x}{1+x} < p + (1-p)0 = v(B)$$

This means that, in the long run, natural selection will lead to a higher survival rate of players choosing option B, reducing the proportion of players choosing A in the population. This pushes parameter p towards $\frac{x}{1+x}$.

Similarly, if $p < \frac{x}{1+x}$, then

$$v(A) = p0 + (1-p)x > \frac{x}{1+x} > p + (1-p)0 = v(B)$$

which would increase the proportion of players choosing A in the long run.

Exercise 7: (Final 2019, 30 points) *Nash Equilibrium.* In a tennis match, each time a player serves the ball, she can choose whether to aim the ball to the right or to the left of her opponent. The receiving player must also decide whether to be more prepared for a right-directed ball or for a left-directed ball the very moment the server makes her serve, since the ball travels very fast. The probability that the server wins the point depends on the strategies chosen by the two players. Assuming that each player chooses her strategy to maximize the probability that she wins the point, we can write the payoff matrix for their interaction as below:

		Receiver	
		Right	Left
Server	Right	$0.1, 0.9$	$0.7, 0.3$
	Left	$0.8, 0.2$	$0.4, 0.6$

Find all the Nash equilibria of this game (in pure or mixed strategies). If the players are indeed playing a Nash equilibrium, what is the probability that the server wins the point?

Answer: This exercise is taken from Aviad Heifetz's Game Theory book and it is based on Walker and Wooders (2001, AER). No equilibrium in pure strategies. As in the matching pennies and penalty kick games, one player wants to do the same thing as the other and the other player wants to do the opposite of it.

To find the equilibria in mixed strategies, let p be the probability that server serves Right and q be the probability that receiver prepares for Right. Server's expected payoff from the her two pure strategies:

$$\begin{aligned} v_s(R,q) &= q(0.1) + (1-q)(0.7) = 0.7 - (0.6)q \\ v_s(L,q) &= q(0.8) + (1-q)(0.4) = 0.4 + (0.4)q \end{aligned}$$

Accordingly, the server's best response function is such that she serves Right if q is low and serves Left if q is high. To find the level of q that makes her indifferent, we equate the expected payoffs above: $q = 0.3$.

Receiver's expected payoff from the her two pure strategies:

$$\begin{aligned} v_r(R,p) &= p(0.9) + (1-p)(0.2) = 0.2 + (0.7)q \\ v_r(L,p) &= p(0.3) + (1-p)(0.6) = 0.6 - (0.3)q \end{aligned}$$

Accordingly, the receiver's best response function is such that she prepares for Right if p is high and prepares for Left if p is low. To find the level of p that makes her indifferent, we equate the expected payoffs above: $p = 0.4$.

When we draw these best response functions, we see that their unique intersection point is at $p = 0.4, q = 0.3$. This is the unique Nash equilibrium of this game: The server serves right 40% probability and the receiver prepares for a Right serve with 30% probability.

To find the probability of winning for the server, we just look at her expected equilibrium payoff from serving Right and serving Left. If we have done our calculations right, these two payoffs must be the same:

$$\begin{aligned} v_s(R,0.3) &= (0.3)(0.1) + (0.7)(0.7) = 0.52 \\ v_s(L,q) &= (0.3)(0.8) + (0.7)(0.4) = 0.52 \end{aligned}$$

Since the game is zero-sum, the receiver's probability of winning the point is 0.48.

Exercise 8: (Midterm 2020, 25 points) Consider the following case study from 1970s: Polaroid is the market leader for instant photography. Kodak will choose whether to enter *in* the instant photography market or to stay *out*. Polaroid either *accommodates* or *fights* a potential entry. The payoffs are given by the following payoff matrix:

Polaroid

Kodak

	Acc	*Fight*
In	2, 1	0, 0
Out	1, 2	1, 2

(5) a) Find the *strictly* and *weakly* dominated pure strategies for both firms.

No strictly dominated strategy for the players. For Polaroid, Fight is weakly dominated by Accommodate.

(10) b) Find all the *Nash equilibria* of the game (in pure and mixed strategies).

We can find the Nash equilibria in pure strategies simply through circling the payoffs: (In, Acc) and $(Out, Fight)$ are the two pure-strategy Nash equilibria. The safest way to find all the Nash equilibria in mixed strategies is drawing the best response correspondences. Let's say p is the probability that player Kodak plays In and q is the probability that Polaroid plays Acc. Best response for Kodak is similar to the best response function in the matching pennies or the battle of the sexes games:

$$p = \begin{cases} 0 & \text{if } q < 1/2 \\ [0,1] & \text{if } q = 1/2 \\ 1 & \text{if } q > 1/2 \end{cases}$$

Best response for Polaroid is rather non-standard. Notice that playing the weakly dominant strategy of Acc is always a part of the best response. When Kodak plays In with some positive probability (when $p > 0$), Acc is the only best response. But when Polaroid is sure that Kodak is playing Out ($p = 0$), it is indifferent between Acc and $Fight$. Accordingly, Polaroid's best response is

$$q = \begin{cases} [0,1] & \text{if } p = 0 \\ 1 & \text{if } p > 0 \end{cases}$$

Drawing these two correspondences on the $p \times q$ coordinate system identifies a continuum of mixed strategy Nash equilibria: $\{(p, q) : p = 0 \text{ and } 0 \leq q \leq 1/2\}$ and $(p = 1, q = 1)$. In the first subset of equilibria, Kodak stays out for sure and Polaroid accommodates with some probability q, where q can be any number between 0 and 1/2 (including 0 and 1/2). Notice that one of the pure-strategy Nash equilibria $(Out, Fight)$ is an element of this continuum. The second subset corresponds to the other pure-strategy Nash equilibrium: (In, Acc).

(10) c) *Trembling-Hand-Perfect* Nash Equilibrium: Consider a modification of the game where the two players are not allowed to follow pure strategies: Suppose each player is required to choose a mixed strategy where probability of playing either pure strategy has to be at least ε, where ε is a strictly positive but very small number. Find the set of Nash equilibria of this modified game. What happens to this set as ε converges to zero?

Now the players can choose their mixed strategies p and q from set $[\varepsilon, 1 - \varepsilon]$ instead of set $[0, 1]$. Best response functions should be updated accordingly. For Kodak, this would require changing best responses 0 and 1 with the closest allowed mixed strategies ε and $1-\varepsilon$:

$$p = \begin{cases} \varepsilon & \text{if } q < 1/2 \\ [\varepsilon, 1-\varepsilon] & \text{if } q = 1/2 \\ 1-\varepsilon & \text{if } q > 1/2 \end{cases}$$

For Polaroid, the modification to the game breaks the potential tie between the options of accommodation and fight: Against any allowed mixed strategy of Kodak, the unique best response of Polaroid is maximizing the possibility of accommodation:

$$q = 1-\varepsilon \text{ for all } p \in [\varepsilon, 1-\varepsilon]$$

Accordingly, the unique Nash equilibrium of the modified game is $(p = 1-\varepsilon, q = 1-\varepsilon)$, which converges to $(p = 1, q = 1)$ or (In, Acc) as ε converges to zero.

Remark: (In, Acc) is the unique Trembling-Hand-Perfect Nash Equilibrium (due to Selten) of the original game. The underlying idea for trembling hand perfection is the possibility that players can make mistakes. $Fight$ is a best response for Polaroid only when Polaroid is sure that Kodak will choose *out*. But when Polaroid considers the possibility that Kodak could make a mistake with a small probability and enter in, it will be reluctant to choose $fight$. Weakly dominated strategies are never played in trembling hand perfect equilibria.

We will use this entry game as a motivating example for the introduction of a solution concept that we will use for sequential games, where decisions are made in a sequence (rather than simultaneously).

Exercise 9: (Final 2020, 15 points) Players 1, 2, and 3 face the risk of catching a new viral disease. The disease will be avoided by all if *at least two* out of the three players follow the rules of "social distancing." Avoiding the disease in this way increases the payoff of each of the three players by 100 units. The individual cost of following social distancing for a player is 18 units of payoff. Consider the simultaneous move game where each of the three players simultaneously decides whether to respect the rules of social distancing or not.

(3) a) Are there any *dominated* (weakly or strictly) strategies in this game? Either tell what these strategies are, or explain briefly why they do not exist.

Neither of the two strategies (follow social distancing, do not follow social distancing) is a dominated strategy. Social distancing is a best response if only one of the other players is doing the same, and no social distancing is a best response if both of the other players are following social distancing (or if both of them are not following it).

(5) b) Find all the *pure-strategy Nash equilibria* of this game.

There are 4 such equilibria. A) None of them follow social distancing. (Notice that there is no profitable deviation. They all get sick, but none of them can change the outcome with a unilateral deviation. B) Players 1 and 2 follow social distancing and 3 does not. C) 1 and 3 follow, 2 does not. D) 2 and 3 follow, 1 does not.

The rest of the question is about construction of an equilibrium in *completely mixed strategies*. Suppose each of the players 2 and 3 obeys social distancing with probability p.

(2) c) What is player 1's expected payoff (as a function of p) if he does not follow social distancing measures?

$$p^2 100 + 2p(1-p)\,0 + (1-p)^2\,0$$

(3) d) What are the values of p that will make player 1 indifferent between following social distancing or not following it?

$$\begin{aligned} p^2 100 &= p^2 100 + 2p(1-p)\,100 + (1-p)^2\,0 - 18 \\ 18 &= 2p(1-p)\,100 \\ p(1-p) &= 0.09 \end{aligned}$$

Two solutions to the last equation: $p = 0.9$ and $p = 0.1$.

(2) e) In light of your answer to the question above, find the *symmetric mixed-strategy equilibria* of this game.

Everybody following social distancing with probability $p = 0.9$ is a Nash equilibrium. Everybody following it with probability $p = 0.1$ is another one. (For completeness, notice also that the pure-strategy equilibrium A where none of the players observe social distancing is also a symmetric Nash equilibrium of this game.)