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Homework on Nash Equilibrium
Sketch of the suggested solutions

(20 points) **Exercise 1:** (Midterm 2017) Describe the features (options available to individuals and their preferences) of the Hawk-Dove game we studied in class. You can use a payoff matrix if you find it more convenient. Find a real-life situation that fits to this description.

Solution 1:

	$Hawk$	$Dove$
$Hawk$	$0, 0$	$3, 1$
$Dove$	$1, 3$	$2, 2$

The actual payoffs are not the important part. The defining feature of the Hawk-Dove game is the following ranking of the outcomes from either player's perspective: (being a hawk against a dove) $\succ$ (being a dove against a dove) $\succ$ (being a dove against a hawk) $\succ$ (being a hawk against a hawk). Recall from your economics courses that the symbol ($\succ$) stands for "preferred to."

Real-life situation: You can pick an example from interpersonal or international relations. For instance, two countries are having a dispute over a small territory. Both prefer to claim the territory for itself but neither prefers going to war over it. Will a country choose to take a *hawkish* position and face escalation of the dispute? Such a position will be successful in gaining the territory if the other one chooses to act *dovish*. On the other hand, if the other one assumes an aggressive position as well, then war is inevitable.

(25) **Exercise 2:** (Midterm 2017) A committee of three members must decide on one of the two policy alternatives: A or B. Two of the committee members prefer that A is the chosen policy and the other one prefers B. Each of them will vote individually on either A or B. The policy alternative that receives more votes will be implemented.

(15) a) Find the *Nash equilibria* of the game described above.

NB: Recall that in a Nash equilibrium no player has a profitable deviation.

(5) b) Is there a *strictly dominant* strategy for any of the players? If yes, which one?

(5) c) Is there a *weakly dominant* strategy for any of the players? If yes, which one?

Solution 2: a) Best way to start solving the question is identifying the *strategies* for each player and the *strategy profiles* in the game. Each committee member can vote for either A or B. So each of the three players has two strategies. Thus the number of pure strategy profiles is $2 \times 2 \times 2 = 8$. To find the Nash equilibria of the game, we can think of these 8 profiles one by one and see if any of the players has a profitable deviation:

I start with the obvious equilibrium. AAB: Each voter is voting for its favorite alternative. Policy A is chosen in the end. Players 1 and 2 are happy. Player 3 is not. But is there anything that Player 3 can do about this? Changing Player 3's vote does not change the outcome. So AAB is a Nash Eq.

What about AAA? The outcome of the election is still A. Player 3 is not happy, but changing his vote would not change anything. So AAA is also an equilibrium.

An even more surprising equilibrium is BBB. Players 1 and 2 are not happy with the result of the election (which is B). But do they have an individual deviation that would change the outcome? NO. Put yourself into the shoes of Player 1. Even if you change your vote from B to A, you cannot change the result. So it is not a profitable deviation for this player. Same for Player 2. You may ask why don't players 1 and 2 *both* change their votes collectively? But notice that the players of the game are the individual committee members, not coalitions of them.

There is no other strategy profile which would be an equilibrium. You should try writing them one by one and noticing the profitable deviations.

For instance, ABB is not equilibrium. Because, Player 2 can improve the outcome from her perspective, by changing her vote from B to A.

Alternatively, to identify the best response functions and to find the Nash equilibria, you can write down the payoff table for the game - where one player chooses between the rows, the other between the columns, and the last one between the matrices.

b) There is no strictly dominant strategy for any of the players. Voting for your preferred alternative gives you strictly higher payoff only when your vote is *pivotal* - when you can change the outcome with your vote.

c) For each player, voting for the preferred alternative is a weakly dominant strategy: For Players 1 and 2, A weakly dominates B; for Player 3, B weakly dominates A.

(20 points) **Exercise 3:** (Midterm 2018) Consider the following normal-form game, where the pure strategies for Player 1 are U, M, and D, and the pure strategies for Player 2 are L, C, and R. The first payoff in each cell of the matrix belongs to Player 1, and the second one belongs to Player 2.

Player 2

Player 1		L	C	R
	U	$6,8$	$2,6$	$8,2$
	M	$8,2$	$4,4$	$9,5$
	D	$8,10$	$4,6$	$6,7$

(7) a) Find the *strictly dominated* strategies for each of the players. Make sure to write down which strategies strictly dominate them.

(7) b) Which strategies survive the process of *iterated elimination* of strictly dominated strategies?

(6) c) Find the Nash equilibria of this game.

Solution 3:

a) For player 1, U is strictly dominated by M. For player 2, there is no strictly dominated strategy.

b) Once strategy U is removed for player 1, in the remaining 2x3 game, C is strictly dominated by R for player 2. After removing C, nothing else is dominated in the remaining 2x2 game.

c) (D,L) and (M,R) are the Nash equilibria of this game.

Exercise 4: (Final 2018) The United States is deciding on the magnitude of the protectionist trade policies that it will implement. Suppose that the level of such policies can be represented by the non-negative number a_1. Simultaneously, the European Union is making a similar choice and setting its own policies $a_2 > 0$. The payoff of each player i is given by $v_i(a_1, a_2) = a_i - 2a_j + a_i a_j - (a_i)^2$.

a) Find a Nash equilibrium of this game. Is it unique?

b) Is the equilibrium outcome you found in the previous part Pareto efficient?

c) If the players could sign a binding trade agreement on the levels of protectionist policies that they could implement, what levels would they choose?

Solution 4:

a) This is just a different scenario for Exercise 5.14 in Tadelis's textbook.

$$\begin{aligned} \frac{dv_i(a_1, a_2)}{da_i} &= 1 + a_j - 2a_i = 0 \\ a_i &= \frac{1 + a_j}{2} \end{aligned}$$

Solving $a_1 = \frac{1+a_2}{2}$ and $a_2 = \frac{1+a_1}{2}$ together gives $a_1 = a_2 = 1$. The Nash equilibrium is unique.

b) The equilibrium payoff is $v_i = 1 - 2 + 1 - 1 = -1$ for each player. If the players continue choosing the same policy level a, the payoff of each of them would be $v_i(a, a) = a - 2a + a^2 - a^2 = -a$, which is decreasing in a. So by setting a lower level of a than 1, we can improve the payoffs of both players. This implies that the equilibrium outcome is not Pareto efficient.

c) The joint surplus would be maximized by setting $a_1 = a_2 = 0$.

Exercise 5: (Final 2019, 30 points) *Dominated Strategies.* Consider the following Cournot duopoly game. Each of the two players will choose a non-negative real number (q_1, q_2) and the resulting payoffs will be $v_1(q_1, q_2) = (120 - q_1 - q_2)q_1$ and $v_2(q_1, q_2) = (120 - q_1 - q_2)q_2$.

(10) a) Show that strategy $q_i = 60$ strictly dominates any strategy larger than 60 for player i.

(10) b) Once you eliminate the strictly dominated strategies, you transform the game into a reduced game where players choose their strategies from set $[0, 60]$. In this reduced game, show that any value of q_i smaller than 30 is strictly dominated for player i.

(10) c) What you have showed above are the first two steps of the process of "iterated elimination of strictly dominated strategies." Suppose we know that this process will eliminate all the strategies except one. Find the strategy that would survive this iterated elimination process. You can use the property that if an iterated-elimination equilibrium exists, it is also a Nash equilibrium.

Solution 5:

a)

$$v_1(60, q_2) = (60 - q_2)\,60$$

We need to show that this is strictly larger than $(120 - q_1 - q_2)\,q_1$ for $q_1 > 60$ and for any non-negative value of q_2.

$$\begin{aligned} (60 - q_2)\,60 &> (120 - q_1 - q_2)\,q_1 \\ 60^2 + (q_1)^2 - 120\,(q_1) &> q_2\,(60 - q_1) \end{aligned}$$

The inequality holds since the left-hand-side (equaling $(60 - q_1)^2$ is strictly positive and the right-hand-side is strictly negative for $q_1 > 60$.

b) Consider:

$$v_1(30, q_2) = (90 - q_2)\,30.$$

If we show that this is strictly larger than $(120 - q_1 - q_2)\,q_1$ for $q_1 < 30$ and $q_2 \in [0, 60]$ we know that $q_1 < 30$ is dominated.

$$\begin{aligned} (90 - q_2)\,30 &> (120 - q_1 - q_2)\,q_1 \\ (90)\,(30) + (q_1)^2 - 120\,(q_1) &> q_2\,(30 - q_1) \end{aligned}$$

Since $q_1 < 30$, the right-hand-side is increasing in q_2. For the inequality to hold for all values of $q_2 \in [0, 60]$, it must hold for its highest possible value 60:

$$\begin{aligned} (90)\,(30) + (q_1)^2 - 120\,(q_1) &> 60\,(30 - q_1) \\ (30)^2 + (q_1)^2 - 60\,(q_1) &> 0 \\ (30 - q_1)^2 &> 0 \end{aligned}$$

This is obviously true.

c) We basically need to find the N.E. of the game. The first order condition to maximize $v_1(q_1, q_2)$ by choosing q_1 is

$$120 - 2q_1 - q_2 = 0.$$

Symmetrically, the first order condition to maximize $v_2(q_1, q_2)$ by choosing q_2 is

$$120 - q_1 - 2q_2 = 0.$$

Solving these two equations together, we find the unique Nash equilibrium as $q_1 = q_2 = 40$.

Exercise 6: (Midterm 2020, 20 points) Consider the following variation on the tragedy of commons game: Two herders simultaneously decide the number of animals to graze in the common field (a_1 and a_2). The per animal value of grazing is $200 - (a_1 + a_2)^2$. The payoff of each herder i is the total value of his herd $a_i \left[200 - (a_i + a_j)^2\right]$.

(10) a) Find the Nash equilibrium of this game. (*Hint*: you do not need to find the exact solutions for the herders' best-response functions.)

(10) b) Is the Nash equilibrium outcome *Pareto efficient*?

Solution 6:

a) We start with the optimization problem of player 1:

$$\max_{a_i} a_i \left[200 - (a_i + a_j)^2\right]$$

The first-order condition for maximization is

$$\begin{aligned} 200 - (a_i + a_j)^2 - 2a_i(a_i + a_j) &= 0 \\ 200 - (3a_i + a_j)(a_i + a_j) &= 0 \end{aligned}$$

Solving this equation would give a_i as a best response to a_j. In a Nash equilibrium, each player plays a best response to the other player's choice of play. So the above equation must be satisfied for both $i = 1$ and $i = 2$ for the Nash equilibrium levels of a_1 and a_2:

$$\begin{aligned} (3a_1 + a_2)(a_1 + a_2) &= 200 \\ (a_1 + 3a_2)(a_1 + a_2) &= 200 \end{aligned}$$

Now you can use the symmetry of the game and assume a symmetric equilibrium such that $a_1 = a_2$. Then the equation turns into $8a_1^2 = 200 \Rightarrow a_1^2 = 25 \Rightarrow a_1 = a_2 = 5$. This is the Nash equilibrium. **Remark**: You can solve the above system of equation without assuming symmetry in fact. To see this, add up the two equations above: $4(a_1 + a_2)(a_1 + a_2) = 400 \Rightarrow (a_1 + a_2) = 10$. Substitute in the first equation to get

$$(2a_1 + 10)\,10 = 200 \Rightarrow a_1 = 5 \text{ and } a_2 = 5$$

b) The payoff of each herder under the equilibrium outcome is $a_i \left[200 - (a_i + a_j)^2\right] = 5\left[200 - 10^2\right] = 500$. If they had 4 animals each, their payoff would have been $4\left[200 - 8^2\right] = 4 \times 136 = 544$. So there is an alternative way to play the same game for the herders and receive a higher payoff. Therefore the equilibrium outcome is not Pareto efficient.