Dimitrios Zormpas

Homework on Dominated Strategies

Sketch of the Suggested Solutions

**Exercise 1:** (Midterm 2019, 25 points) The following paragraph is taken from the article "Coping with Asymmetries in the Commons" by Elinor Ostrom and Roy Gardner, *Journal of Economic Perspectives*, 1993.

"We model the strategic interaction between headenders (player 1) and tailenders (player 2) of an irrigation system as follows. There is a temporary structure, a headworks, at the very beginning of the system, which brings water into the system. This structure has to be rebuilt annually. The total amount of water ($W$) brought into the system depends on how much labor headenders ($L_1$) and tailenders ($L_2$) provide. The decision to provide labor is taken simultaneously under the condition of complete information. Once water starts flowing, headenders, who get first crack at it, take the lion's share of it (75 percent), while tailenders get what is left (25 percent). The opportunity cost of providing a unit of labor is constant throughout the system, and equal to 1."

Later in the same paper, as a numerical example, the authors consider the following *water production function*:

$$W = 2\left(L_1^{0.5} + L_2^{0.5}\right).$$

(10) a) Assume that each player maximizes its own share of the water minus the opportunity cost of the labor that it provides. Find the strictly dominant strategy equilibrium in this game. What is the equilibrium level of water that is brought into the system? What are the equilibrium payoffs?

NB: Approach this as a utility maximization problem for each player.

(5) b) If the players could sign a binding agreement on the levels of the labor that they would provide, what would be the efficient levels that maximize the total amount of water net of the opportunity cost of labor? How much water would be brought to the system in this case?

NB: Approach this as a joint utility maximization problem.

(5) c) What is the lowest amount of water that each of the players would demand in order to accept to provide the labor levels in the agreement that you found in part (b), instead of following the equilibrium strategies that you found in part (a)?

(5) d) In light of your earlier findings, is the strictly dominant strategy equilibrium outcome Pareto efficient? Why or why not?

a) The utility maximization problem that the headenders face is:

$$\max_{L_1} U_1(L_1, L_2) = \max_{L_1} \frac{3}{4} W(L_1, L_2) - L_1$$

The first order condition gives:

$$\begin{aligned} U_1'(L_1^*, L_2) &= 0 \\ \frac{3}{4} L_1^{*0.5-1} - 1 &= 0 \\ L_1^{*0.5} &= \frac{3}{4} \\ L_1^* &= \frac{9}{16} \end{aligned}$$

Similarly, the utility maximization problem that the tailenders face is:

$$\max_{L_2} U_2(L_1, L_2) = \max_{L_2} \frac{1}{4} W(L_1, L_2) - L_2$$

The first order condition gives:

$$\begin{aligned} U_2'(L_1, L_2^*) &= 0 \\ \frac{1}{4} L_2^{*0.5-1} - 1 &= 0 \\ L_2^* &= \frac{1}{16} \end{aligned}$$

Since $L_1^*, L_2^*$ are the global maxima for the two groups, they are by construction the dominant strategies. From these we obtain also the equilibrium level of water that is brought into the system $W(L_1^*, L_2^*) = 2\left(\frac{3}{4} + \frac{1}{4}\right) = 2$, and the corresponding payoffs $U_1(L_1^*, L_2^*) = \frac{6}{4} - \frac{9}{16} = \frac{15}{16}$ and $U_2(L_1^*, L_2^*) = \frac{2}{4} - \frac{1}{16} = \frac{7}{16}$.

**NB: You can easily verify that the problem is well posed (the objective is a concave function).**

b) Now the problem that we need to solve is as follows:

$$\begin{aligned} & \max_{L_1, L_2} U_1(L_1, L_2) + U_2(L_1, L_2) \\ = \ & \max_{L_1, L_2} W(L_1, L_2) - L_1 - L_2 \end{aligned}$$

The first order conditions give:

$$\begin{aligned} L_1^{**} &= 1 \\ L_2^{**} &= 1 \end{aligned}$$

In this case, $W(L_1^{**}, L_2^{**}) = 2(1+1) = 4$.

c) The headenders would prefer the latter allocation as long as $U_1^{**} > U_1(L_1^*, L_2^*) \rightarrow W_1 - 1 > \frac{15}{16} \rightarrow W_1 > \frac{31}{16}$. Similarly, for the tailenders we have $U_2^{**} > U_2(L_1^*, L_2^*) \rightarrow W_2 - 1 > \frac{7}{16} \rightarrow W_2 > \frac{23}{16}$.

d) If we can find a labour allocation different from $(L_1^*, L_2^*)$ that makes one player better-off without making the other worse-off then $(L_1^*, L_2^*)$ is not a Pareto efficient outcome. In fact, the allocation $(L_1^{**}, L_2^{**})$ guarantees $4\left(= \frac{64}{16}\right)$ units of water that are more than enough to cover the water needed to make both groups better off $\left(W_1 + W_2 = \frac{31}{16} + \frac{23}{16} = \frac{54}{16} < \frac{64}{16}\right)$. Hence $(L_1^*, L_2^*)$ is not a Pareto efficient outcome.

**Exercise 2:** (Midterm 2020, 25 points) An item is up for auction. Player 1 values the item at 3 euros while Player 2 values it at 5 euros. Each player can bid either 0, 1, or 2 (in euros). If player $i$ bids more than player $j$ then $i$ wins the good and pays his bid, while the loser does not pay. If both players bid the same amount then a fair coin is tossed to determine who the winner is, and the winner gets the good and pays his bid while the loser pays nothing. Assume that players are risk neutral.

(5) a) Write down the game in matrix form.

(10) b) Does any player have a *strictly dominated strategy*?

(10) c) Consider the following procedure of *iterated elimination of strictly dominated strategies* (IESDS). Remove the strictly dominated strategies that you found above. Now you have a new game with a smaller number of strategies for the players. Find if there are strictly dominated strategies in this new game. Continue in this fashion. Can you make a prediction on which strategies will be chosen by the two players?

a) Each player can bid 0, 1, or 2 euros so the payoff matrix will be 3*3. Unless there is a tie, the payoffs are straightforward and equal the difference "value-bid". If there is a tie, a fair coin is tossed so there is a 50-50 chance for the player to be the winner/loser. The winner gets "value-bid" and the loser gets zero. Hence:

| player 1/player 2 | bid 0 | bid 1 | bid 2 |
|---|---|---|---|
| bid 0 | $\frac{1}{2}(3-0)+\frac{1}{2}0, \frac{1}{2}(5-0)+\frac{1}{2}0$ | $0,(5-1)$ | $0,(5-2)$ |
| bid 1 | $(3-1),0$ | $\frac{1}{2}(3-1)+\frac{1}{2}0, \frac{1}{2}(5-1)+\frac{1}{2}0$ | $0,(5-2)$ |
| bid 2 | $(3-2),0$ | $(3-2),0$ | $\frac{1}{2}(3-2)+\frac{1}{2}0, \frac{1}{2}(5-2)+\frac{1}{2}0$ |

Alternatively the table looks as follows:

| player 1/player 2 | bid 0 | bid 1 | bid 2 |
|---|---|---|---|
| bid 0 | $\frac{3}{2}, \frac{5}{2}$ | $0,4$ | $0,3$ |
| bid 1 | $2,0$ | $1,2$ | $0,3$ |
| bid 2 | $1,0$ | $1,0$ | $\frac{1}{2}, \frac{3}{2}$ |

b) Checking carefully we have the following:

| player 1/player 2 | bid 0 | bid 1 | bid 2 |
|---|---|---|---|
| bid 0 | $\frac{3}{2}, \frac{5}{2}$ | $0,\underline{4}$ | $0,3$ |
| bid 1 | $\underline{2},0$ | $\underline{1},2$ | $0,\underline{3}$ |
| bid 2 | $1,0$ | $\underline{1},0$ | $\underline{\frac{1}{2}},\underline{\frac{3}{2}}$ |

Bidding 0 is a dominated strategy for both players.

c) After deleting the "bid 0" row and column we have:

| player 1/player 2 | bid 1 | bid 2 |
|---|---|---|
| bid 1 | $1,2$ | $0,\underline{3}$ |
| bid 2 | $1,0$ | $\frac{1}{2},\underline{\frac{3}{2}}$ |

We see that pl. 2 never bids 1 (bid 2 strictly dominates bid 1). Hence the game becomes:

| player 1/player 2 | bid 2 |
|---|---|
| bid 1 | $0,3$ |
| bid 2 | $\underline{\frac{1}{2}}, \frac{3}{2}$ |

In this final form, pl. 1 never bids 1 (bid 2 strictly dominates bid 1). So the strategy profile "bid 2, bid 2" survives IESDS.