Lecture 7

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Overview

- Prisoner's dilemma with two types of P2
- Bayesian game
- Bayesian Nash Equilibrium
- Bayesian Nash Equilibrium in Cournot competition
- Examples

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- **Prisoner's dilemma with two types of P2**
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Return to prisoner's dilemma

- Up to now, we assumed that the types are common knowledge.
- It is highly likely that you do not know if P2 is the chief's brother or not...
- P2 can be of type a:

$v1/v2$	D	C
D	$0, -2$	$-5, -1$
C	$-1, -5$	$-4, -4$

- or of type b:

$v1/v2$	D	C
D	$0, -2$	$-5, -3$
C	$-1, -5$	$-4, -6$

Return to prisoner's dilemma (cont.)

- Player 2 type b receives an additional jail term of 2 years in case that she confesses.
- Player 2 knows her type. But player 1 does not know it. Player 1 assigns the probability p to the possibility that Player 2 is type a.
- To talk about the equilibrium, we need to specify how the different types of different players will play the game.

Return to prisoner's dilemma (cont.)

- Easy to see that Player 2 type a plays confess and type b plays deny.
- What about Player 1?
- $v(deny) = p(-5) + (1-p)0 = -5p$
- $v(confess) = p(-4) + (1-p)(-1) = -3p - 1$
- If $p > 1/2$ then Player 1 plays confess.
- If $p < 1/2$ then Player 1 plays deny

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Bayesian game

- Each player in a game may be one of a number of possible types.
- Each player observes her own type and has a belief (probability distribution) on the other players' types.
- A player's payoff may depend on
  - her actions and the other players' actions (as before), and
  - her type and the other players' types (specific for Bayesian games).

Bayesian game: components

- Players $i \in N$
- Actions $a_i \in A_i \forall i$
- Types $\theta_i \in \Theta_i \forall i$
- Payoff function $v_i\left(a_i, a_{-i}, \theta_i, \theta_{-i}\right) \forall i$
- Probability distribution p over the types of the players.
- We can consider a Bayesian game as a game where each type of each players acts as a different player.

Bayesian game: components

- types can be independently distributed
  - probability that player i has type θ_i is $p_i\left(\theta_i\right)$
  - probability that player j has type θ_j is $p_j\left(\theta_j\right)$
  - $p_i\left(\theta_i, \theta_j\right) = p_i\left(\theta_i\right) p_j\left(\theta_j\right)$
- Alternatively, different players' types may be **correlated** in some way.

Bayesian game: components

- Each player knows her type
- Each player's action can be conditioned on her type
- Player i's Bayesian strategy is a function from the type set Θ_i into the action set A_i: $s_i(\theta_i)$.

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Bayesian Nash Equilibrium

- A Bayesian strategy profile $s^* = (s_1^*, s_2^*, ...s_n^*)$ is a Bayesian Nash Equilibrium if:

$$E_{\theta_{-i}} v_i \left(s_i^* (\theta_i), s_{-i}^* (\theta_{-i}), \theta_i, \theta_{-i}\right) \geq E_{\theta_{-i}} v_i \left(s_i' (\theta_i), s_{-i}^* (\theta_{-i}), \theta_i, \theta_{-i}\right) \quad \forall s_i' (\theta_i), \theta_i, i$$

- $E_{\theta_{-i}}$ means expectation of player i on the types of the other players.
- We must use expected payoffs because each player's payoff depends on the types of the other players
- We still have best responses
- No type of no player has a profitable deviation.

Bayesian Nash Equilibrium

- We obtained:
  - If $p > 1/2$ then Player 1 plays confess.
  - If $p < 1/2$ then Player 1 plays deny
- Suppose that probability that P2 is type a is $p = 2/3$
- P2 type a: confess
- P2 type b: deny
- P1: confess

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Bayesian Nash Equilibrium: Cournot

- Suppose two quantity-setting firms compete in a market for which the inverse demand is given as $P = 100 - q_1 - q_2$.
- Suppose further that per-unit production costs are as follows: $c_1 = c_2 = 10$.
- We found the Nash equilibrium of this game as $q_1 = q_2 = 30$ and the equilibrium payoffs as $v_1 = v_2 = 900$.

Bayesian Nash Equilibrium: Cournot

- Now suppose that $c_1 = 10$, but c_2 may be either high $(= 16)$ or low $(= 4)$
- Player 1 assigns equal probabilities to these two "types" for Player 2, so that the expected $c_2 = 10$.
- Player 2 does not have to consider expectations because it knows that there is only a single type for Player 1.

Bayesian Nash Equilibrium: Cournot

- Player 2 chooses q_2 to maximize $v_2 = (P - c_2)q_2 = (100 - c_2 - q_1 - q_2)q_2$
- The first-order condition for a maximum is

$$q_2^* = (100 - c_2 - q_1)/2$$

- Depending on c_2, this is either

$$q_{2H}^* = (84 - q_1)/2$$

or

$$q_{2L}^* = (96 - q_1)/2$$

- These are the best response functions of the two types of Player 2.

Bayesian Nash Equilibrium: Cournot

- Player 1 must take into account that Player 2 could have either high or low unit costs so its expected payoff is

$$
\begin{aligned}
Ev_1 &= 0.5(100 - c_1 - q_1 - q_{2H}^*)q_1 + 0.5(100 - c_1 - q_1 - q_{2L}^*)q_1 \\
&= \left[100 - c_1 - q_1 - 0.5q_{2H}^* - 0.5q_{2L}^*\right] q_1 \\
&= \left[90 - q_1 - 0.5q_{2H}^* - 0.5q_{2L}^*\right] q_1
\end{aligned}
$$

- The first-order condition for a maximum gives us the best response of 1 as a function of q_{2H}^* and q_{2L}^*

$$q_1^* = (90 - 0.5q_{2H}^* - 0.5q_{2L}^*)/2$$

Bayesian Nash Equilibrium: Cournot

- Best response functions give a system of three equations and three unknowns:

$$
\begin{aligned}
q_{2H}^* &= (84 - q_1)/2 \\
q_{2L}^* &= (96 - q_1)/2 \\
q_1^* &= (90 - 0.5q_{2H}^* - 0.5q_{2L}^*)/2
\end{aligned}
$$

- The Bayesian Nash equilibrium is

$$q_1^* = 30, q_{2H}^* = 27, q_{2L}^* = 33$$

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Study Groups Section (12.2.2) in Textbook

- Working on a group HW
- each student makes an effort
- fruits of the combined efforts are shared by the team

Consider a group of two students:

- Each student decides whether to put effort or not $(e_i = 1/e_i = 0)$
- Effort implies c units of cost
- If either one of the students puts the effort homework is a success, otherwise it is a failure
- Value of the success for each student is private information- known only to this student.

Study Group as a Bayesian Game

- Players 1 and 2
- Actions: $e_i \in \{0, 1\}$
- Types of player i: $\theta_i \in [0, 1]$
  - types are uniformly and independently distributed
  - cumulative distribution function $F_i = prob\,(\theta_i \leq x)$
  - the cost of effort $c \in (0, 1)$
  - Payoff:

$$v_i = \begin{cases} \theta_i^2 - ce_i \text{ if success} \\ 0 \text{ if failure} \end{cases}$$

  - NB: "success" if at least one player puts the effort

One student decision problem

- I am the only one who can exert effort so:

$$v_i = \begin{cases} \theta_i^2 - c \text{ if success} \\ 0 \text{ if failure} \end{cases}$$

- hence the optimal effort decision is:

$$s_i\left(\theta_i\right) = \begin{cases} 1 \text{ if } \theta_i > \sqrt{c} \\ 0 \text{ if } \theta_i < \sqrt{c} \end{cases}$$

Two student decision problem

- I am **not** the only one who can exert effort.
- If I do, I get $\theta_1^2 - c$
- If I do not but the other player does, then I get θ_1^2 without paying a cost.
- The probability that the other player exerts effort, that is, her value from success is high is $prob\left[\theta_2 > \hat{\theta}_2\right]$ where $\hat{\theta}_2$ is a success cutoff that we need to compute:
- $prob\left[\theta_2 > \hat{\theta}_2\right] = 1 - prob\left[\theta_2 \leq \hat{\theta}_2\right] = 1 - \hat{\theta}_2$

Two student decision problem

- I am indifferent between exerting and not exerting effort if

$$\begin{aligned} \theta_1^2 - c &= \left(1 - \hat{\theta}_2\right)\theta_1^2 \\ c &= \hat{\theta}_2\theta_1^2 \end{aligned}$$

- From symmetry: $c = \hat{\theta}_1\theta_2^2$
- Solving these two we obtain: $\hat{\theta}_1 = \hat{\theta}_2 = c^{1/3}$

$$s_i\left(\theta_i\right) = \begin{cases} 1 \text{ if } \theta_i > c^{1/3} \\ 0 \text{ if } \theta_i < c^{1/3} \end{cases} \text{ for } i = 1, 2$$

Battle of the sexes

- wife (P1) can be of type a (loving):

$v1/v2$	F	O
F	$1,3$	$0,0$
O	$0,0$	$3,1$

- or of type b (leaving):

$v1/v2$	F	O
F	$0,3$	$3,0$
O	$3,0$	$0,1$

Battle of the sexes

- Assumptions:
  - The wife knows her preferences. She has two types "loving" or "leaving.
  - The husband does not know his wife's real preferences; he attaches a probability p to the fact that she is "loving" and $1-p$ to the fact that she is leaving.
  - The wife knows her husband's estimates of her preferences, i.e., she knows the value of p.

Battle of the sexes

- contrary to the prisoner's dilemma:
  - there are no dominated strategies for the two-type player
  - the two-type player is the row player
- How to approach this:
- Step 1: Suppose that the column player plays F for sure. In a best response
  - the row player chooses F when of type a
  - the row player chooses O when of type b
- Step 2: Under what condition this was a good move for the column player?
  - By playing F I get $3 * p + 0 * (1 - p) = 3p$
  - By playing O I get $0 * p + 1 * (1 - p) = 1 - p$
- So if $3p \geq 1 - p \rightarrow p \geq 1/4$, F is a best response to (F,O).

Battle of the sexes

- Let's repeat starting the other way around:
- Step 1: Suppose that the column player plays O for sure. In a best response
  - the row player chooses O when of type a
  - the row player chooses F when of type b
- Step 2: Under what condition this was a good move for the column player?
  - By playing O I get $1 * p + 0 * (1 - p) = p$
  - By playing F I get $0 * p + 3 * (1 - p) = 3(1 - p)$
- So if $p \geq 3(1 - p) \rightarrow p \geq 3/4$, O is a best response to (O,F).
- When $p \geq 3/4$ there are two BNE: (F, (F,O)) and (O, (O,F))
- When $3/4 > p \geq 1/4$ there is one BNE: (F, (F,O))
- When $p < 1/4$ there are no BNE in pure strategies

Battle of the sexes

- Solve the prisoner's dilemma following these steps.

Summary

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The end

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