Lecture 5

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Overview

- Extensive vs. Normal form games
- Subgame perfect Nash Equilibrium
- Examples
- Stackelberg competition
- Examples of zero-sum games
- Strategic voting

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Extensive Form versus Normal Form

- The games we studied so far are **normal form games**.
- Normal form games model situations in which players choose strategies without knowing the strategy choices of the other players.
- Three ingredients of the normal form game:
  - Players
  - Strategies
  - Payoffs

Extensive Form versus Normal Form

- In some situations players observe other players' moves before they move.
- Such games can be studied better in **extensive form**.
- Extensive form games provide more information than the normal form games:
  - order of moves
  - actions available at different points in the game
  - information available at different points in the game
- Easiest way to represent an extensive form game is to use a **game tree**.

Example 1: An Entry Game

- Year 1976: Kodak is contemplating entering the instant photography market.
- Polaroid is the market **leader** (monopolist) since 1950s.
- If Kodak enters, Polaroid can either **fight** the entry or **accommodate** it.
- game tree

Entry Game, the game tree

What is in a game tree?

- Nodes (histories)
  - Decision nodes
  - Terminal nodes
- Branches
  - Players
  - Actions
- Payoffs
- Information sets (to be added later)
- Graph...

Entry Game in normal form

- We can represent the entry game in normal form as well.
- A (pure) strategy of a player is a complete contingent plan.
- It should specify an action at each decision node of the player.
- Set of (pure) strategies for Kodak = {in, out}
- Set of strategies for Polaroid ={accommodate, fight}

Entry Game, the payoff matrix

- As we already know, the easiest way to represent the normal form is to use a payoff matrix.
- Make sure you understand how we find the payoffs.
- Matrix...

Entry Game, the payoff matrix

- Find the best response for each player
- There are two pure-strategy N.E.: (In, Accommodate) and (Out, Fight)
- The second equilibrium is rather implausible
  - It implies that, after observing that Kodak has entered the market, Polaroid would have fought even though it is self-destructive to do so.
  - It incorporates a non-credible threat on the part of Polaroid!

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Towards a Reasonable Solution

- Remove the non-credible threats.
- Let Polaroid choose a *sequentially rational* action after observing Kodak's decision.
  - **Polaroid**: *Accommodate*
  - **Kodak**: *In*
- This is a **subgame perfect (Nash) equilibrium**
  - a Nash equilibrium in which the strategy choices of players do not involve non-credible threats.

Subgame Perfect Equilibrium

- A "subgame" is the portion of a larger game that **begins at a decision node corresponding to an information set and includes all future actions stemming from that node**.
- To qualify to be a subgame perfect equilibrium, a strategy profile must give us a Nash equilibrium in each subgame of a larger game.
- Subgame perfect equilibria are identified by the process of **backward induction**.

Entry Game Version 2

- What happens if some players move at the same time?
- Consider the extended version of the entry game.
- After entry, both players can choose accommodate / fight.
- They don't observe each others' moves at the time they decide
- Kodak does not know if Polaroid will accommodate or fight.
- The dashed line is an information set.
- Kodak cannot differentiate between the two nodes on the same information set.
- Graph...

Entry Game Version 2

- Let's find the strategy sets for the two players.
- **A strategy is a complete contingent plan.**
- Each player's strategy should tell her what to do each time she could choose an action.
- Polaroid: Only one time it can decide. Strategy set is {Accommodate, Fight}.
- Kodak: There are two instances it could make a decision.
  - In or Out
  - Accommodate or Fight
- Accordingly, Kodak has 2x2=4 (pure) strategies:
  - {In/Accommodate, In/Fight, Out/Accommodate, Out/Fight}

Entry Game Version 2

- For example, "Out/fight" for Kodak means: "I am choosing to stay out of the market. But if I have found myself in the market somehow, I would have chosen to fight."
- **A strategy is a complete contingent plan.**
- 3 Nash Equilibria:
  - (In/Accommodate, Accommodate)
  - (Out/Accommodate, Fight)
  - (Out/Fight, Fight)

$v1/v2$	A	F
I/A	$2, 1$	$-3, -1$
I/F	$-1, -1$	$-2, -2$
O/A	$1, 2$	$1, 2$
O/F	$1, 2$	$1, 2$

Entry Game Version 2

- But there is only one subgame perfect equilibrium of the Entry Game - version 2.
- To find it, solve for the Nash equilibrium of the "last" subgame.
- Graph...
- In matrix form this is:

$v1/v2$	A	F
A	$2, 1$	$-3, -1$
F	$-1, -1$	$-2, -2$

- Nash equilibrium of the subgame is (Accom., Accom.)
- Accordingly, the unique subgame perfect Nash equilibrium is (In/Accommodate, Accommodate)
- Kodak will never play out (2>1)

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Example: A Two-Period Battle of the Sexes

- Let's recall the game "battle of the sexes."
- Assume that Alex (Player 1) chooses where to spend the evening first and then Chris (Player 2) makes her choice (after observing what Player 1 is doing).
- In effect, that means that the game has become a two-period game
  - Player 2's strategic choices must take into account the information available at the start of period two.
- Game tree

Example: A Two-Period Battle of the Sexes

- Each strategy of Ply 2 is stated as a pair of actions showing what Ply 2 will do depending on Ply 1's actions.
- (o,f) means that Ply 2 goes to Opera if Ply 1 goes to Opera and Ply 2 goes to Football game if Ply 1 goes to Football game.
- (o,o) means that Ply 2 goes to the Opera regardless.

$v1/v2$	o,o	o,f	f,o	f,f
O	2, 1	2, 1	0, 0	0, 0
F	0, 0	1, 2	0, 0	1, 2

- There are 3 pure strategy Nash equilibria in this game {O,oo; O,of; F,ff}.

Example: A Two-Period Battle of the Sexes

- Out of the 3 pure strategy Nash equilibria in this game {O,oo; O,of; F,ff} only (O,of) is a subgame perfect one.
- The other two incorporate non-credible threats on the part of P2.
- Remove the non-credible threats.
- Let Player 2 choose sequentially rational actions.
- Player 2 will play (o,f)
- Player 1 will recognize this and choose O
- subgame perfect equilibrium is Ply 1: O, Ply 2:(o,f)

Curse of Rationality: Centipede Game

- Two players take turns in deciding to terminate the game (N or n) or to continue with it (C or c).
- Tree
- Can you find the unique subgame perfect equilibrium?
- What are the equilibrium payoffs?
- Is the equilibrium outcome Pareto efficient?

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Stackelberg Game

- Recall the Cournot competition game
- Each firm chooses a non-negative real number (quantities)
- Payoffs are:

$$U_1(q_1, q_2) = (90 - q_1 - q_2)\, q_1$$
$$U_2(q_1, q_2) = (90 - q_1 - q_2)\, q_2$$

- **Sequential version:**
- Firm 1 chooses first
- Firm 2 observes q_1 and then chooses q_2
- A strategy for Firm 1 is a non-negative number.
- A strategy for Firm 2 is a function $q_2 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$

Stackelberg Game

- To find the subgame perfect equilibrium strategy of Firm 2, choose q_2 to maximize her payoff given q_1.
- This is basically the best response that we already know: $q_2^*(q_1) = \max\left\{0, \frac{90-q_1}{2}\right\}$.
- This is the equilibrium in each subgame that Firm 1 could generate with its choice of q_1.
- We will use backward induction to find the subgame perfect equilibrium of the game.
- Firm 1 is aware that its choice of q_1 will determine q_2 via the formula $q_2^*(q_1) = \max\left\{0, \frac{90-q_1}{2}\right\}$
- At the first stage of the game, Firm 1 solves the maximization problem:

$$\max_{q_1} U_1\left(q_1, q_2^*(q_1)\right)$$

- Note that this is different from $\max_{q_1} U_1(q_1, q_2)$

Stackelberg Game

$$\begin{aligned} U_1\left(q_1, q_2^*(q_1)\right) &= \left(90 - q_1 - q_2^*(q_1)\right) q_1 \\ &= \left(90 - q_1 - \frac{90 - q_1}{2}\right) q_1 \\ &= \frac{90 - q_1}{2} q_1 \end{aligned}$$

- The FOC results in $q_1^* = 45$
- The subgame perfect Nash equilibrium is $q_1^* = 45; q_2^*(q_1) = \max\left\{0, \frac{90-q_1}{2}\right\}$
- HW: study Stackelberg and Cournot in parallel. Compare per-firm and total quantities and payoffs.

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A closer look to sequential Zero-Sum Games

- Zero-sum games are competitive games where the gain of a player means the loss of another one.
- Example: The 20-dot game that we played the first time:
  - Two players take turns to pick up either 1 or 2 or 3 dots.
  - The player to pick the last dot wins.
- Another example: Matching pennies
- Consider a sequential version of this game:
  - Player 1 puts down his penny either heads up or tails up.
  - Players 2 decides her move after observing Player 1's move.

A closer look to sequential Zero-Sum Games

- Sequential Matching Pennies:
- Game tree...
- Using backward induction, sequentially rational strategy for Player 2 is (t,h).
- Given strategy (t,h) by Player 2, Player 1 is indifferent between the two options
- Two subgame perfect Nash equilibria of this game: H; (t,h) and T; (t,h)
- Both equilibria result with the same outcome: Player 2 wins.

Other Zero-Sum Games

Tic tac toe

- There is a board with 9 positions.
- Two players take turns to place their marks: X and O.
- The first player to get his three marks aligned horizontally, vertically, or diagonally wins.
- What is the subgame-perfect equilibrium outcome of this game?

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Strategic Voting

- Example: Presidential elections in the US
- Consider the three leading candidates in 2020: Joe Biden, Bernie Sanders, and Donald Trump
- Suppose that three groups of voters with approximately equal weights in the electorate.
- Group I: Prefers electing Sanders over Biden, and Biden over Trump. In other words, for Group I, Sanders$\succ$Biden$\succ$Trump

Group II: Biden$\succ$Trump$\succ$Sanders
Group III: Trump$\succ$Sanders$\succ$Biden

Strategic Voting

- In a two-candidate election, any of the candidates can be elected depending on who the adversary is.
- If Sanders and Biden are the two candidates, then Sanders would win with 2/3 of votes.
- If Biden and Trump are the two candidates, then Biden would win with 2/3 of votes.
- If Trump and Sanders are the two candidates, then Trump would win with 2/3 of votes.

Strategic Voting

- Now imagine that the three groups are voting in the democratic party primaries between Biden and Sanders.
- The winner of the primary election will run against Trump in the presidential election.
- If the three groups vote sincerely in the primary election, Sanders receives votes of Groups I and III and advances to the presidential election.
- And Trump wins the presidential election against Sanders.

Strategic Voting as opposed to Sincere Voting

- Now consider Group I voters in the primary.
- By voting for their favorite candidate Sanders, they guarantee that their least preferred candidate Trump is elected.
- Can you see a profitable deviation from sincere voting strategy?
- A vote for Biden (instead of Sanders) by a Group I voter is called a strategic vote.

Strategic Voting

- Take the elections in France as an example.
- Consider three candidates from the Republican (R), Socialist (S), and Macronist (M) parties.
- There are three groups of voters with approximately equal weight in the electorate.
- Group I: Prefers electing R over M, and M over S. In other words, $R \succ M \succ S$
- Group II: $M \succ S \succ R$
- Group III: $S \succ R \succ M$

Strategic Voting

- two-round elections
- Most of the time, two candidates who received the highest number of votes in the first round compete in a second-round election.
- Notice that, any of the three candidates can be elected depending on who the adversary is.
- R and M in the second round: R wins with 2/3 of votes.
- M and S in the second round: M wins with 2/3 of votes.
- S and R in the second round: S wins with 2/3 of votes.

Strategic Voting

- There are elections where one of the candidates is guaranteed to be in the second round, perhaps because of some ideologically motivated voters.
- Suppose candidate S will be in the second round for sure.
- First round is mostly between candidates R and M.
- If the three groups vote sincerely in the first round, candidate R receives votes of Groups I and III and advances to the second round.
- Candidate S wins the election in the second round against candidate R.

Strategic Voting as opposed to Sincere Voting

- Now consider Group I voters in the first round.
- By voting for their favorite candidate R, they guarantee that their least preferred candidate S is elected.
- Can you see a profitable deviation from sincere voting strategy?
- A vote for M (instead of R) by a Group I voter is called a **strategic vote**.

Summary

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The end

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