Lecture 3

Dimitrios Zormpas

CY Cergy Paris Universitè

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Overview

- Nash Equilibrium: Examples
- Cournot duopoly
- Bertrand duopoly
- Hotelling competition
- Revisit the tragedy of the commons

Overview

- **Nash Equilibrium: Examples**
- Cournot duopoly
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Reminder: Best response-Nash Eq.

- A strategy profile $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is a **Nash Equilibrium** if

$$v_i\left(s_i^*, s_{-i}^*\right) \geq v_i\left(s_i', s_{-i}^*\right)$$

- In other words, $s^*$is a **Nash Equilibrium** if all players use their best response functions,

$$s_i^* = b_i\left(s_{-i}^*\right), \forall i$$

Reminder: Best response-Nash Eq.

In a Nash Equilibrium,

- Each player chooses his best strategy given his beliefs about the other players' strategies.
- Moreover, each player's beliefs are correct: They are consistent with what the other players choose.
- This means that, at Nash Equilibrium, no player has an incentive to change his behavior, after learning how the others are behaving.
- No player has a profitable deviation.
- NB: A strictly dominated strategy is never best response...
- ... so if an outcome does not survive IESDS it is not a N.E.

Examples

Find the best responses and identify the N.E.

v_1 / v_2	O	F
O	2, 1	0, 0
F	0, 0	1, 2

Examples

Find the best responses and identify the N.E.

v_1 / v_2	Deny	Confess
Deny	$0, -2$	$-5, -1$
Confess	$-1, -5$	$-4, -4$

Examples

Find the best responses and identify the N.E.

v_1 / v_2	L	C	R
U	4, 3	5, 1	6, 2
M	2, 1	8, 4	3, 6
D	3, 0	9, 6	2, 8

Examples

Find the best responses and identify the N.E.

v_1 / v_2	L	C	R
U	3, 3	5, 1	6, 2
M	4, 1	8, 4	3, 6
D	4, 0	9, 6	6, 8

Examples

Find the best responses and identify the N.E.

v_1 / v_2	Clean	Mess
Clean	2, 2	0, 3
Mess	3, 0	1, 1

Examples

Find the best responses and identify the N.E.

v_1 / v_2	L	C	R
U	6, 6	2, 8	0, 4
M	8, 2	4, 4	1, 3
D	4, 0	3, 1	2, 2

Examples

Find the best responses and identify the N.E.

v_1 / v_2	L	C	R
U	0, 0	−1, 1	1, −1
M	1, −1	0, 0	−1, 1
D	−1, 1	1, −1	0, 0

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Cournot duopoly

- Consider the game with players 1 and 2
- Each player chooses a non-negative real number q
- Payoffs are $v_1(q_1, q_2) = (90 - q_1 - q_2)\, q_1$ and $v_2(q_1, q_2) = (90 - q_1 - q_2)\, q_2$
- Interpretation: q_1 and q_2 production levels of two profit-maximizing firms producing the same good
- The demand function is given by $P(q_1, q_2) = 100 - q_1 - q_2$
- Production cost: $C(q_i) = 10q_i$ for $i \in \{1, 2\}$

Cournot duopoly

- How can we find the best response of player 1?
- Choose q_1 to maximize his payoff given fixed q_2:

$$\max_{q_1} (90 - q_1 - q_2)\, q_1$$

- first-order condition: $q_1^*(q_2) = \frac{90-q_2}{2}$
- Similarly: $q_2^*(q_1) = \frac{90-q_1}{2}$
- Note that these need to be "reasonable" $(90 \geq q_i)$.
- Graph...

Cournot duopoly

- Let's find the N.E.
- Every player chooses its best response hence we need to solve the system:

$$q_1^*(q_2) = \frac{90 - q_2}{2},\, q_2^*(q_1) = \frac{90 - q_1}{2}$$

- $q_1^* = q_2^* = 30$
- What are the payoffs?

$$v_1(q_1^*, q_2^*) = (90 - q_1^* - q_2^*)\, q_1^* = 900$$

- What if they choose 20 units each?

$$v_1(20, 20) = 1000$$

- Is the N.E. a Pareto efficient outcome?
- Since $v_1(20, 20) > v_1(q_1^*, q_2^*)$ it is not.

Cournot n-opoly

- You can show that with n players the best response is $q_i^*\left(q_{-i}\right)=\frac{90-\sum_{j\neq i} q_j}{2}$
- With symmetric players, $q_i=q_1=q_2=\ldots=q_n=q^*$ hence:

$$\begin{aligned} q^* &= \frac{90-(n-1)\,q^*}{2} \\ q^* &= \frac{90}{n+1} \end{aligned}$$

- Then $Q^*=nq^*=90\frac{n}{n+1}$ and $P\left(q_1,\ldots,q_n\right)=100-\sum_{i=1}^{n} q^*=100-90\frac{n}{n+1}$
- We started with $n=2$. Check that for $n\longrightarrow\infty$, P approaches the marginal cost 10.
- NB: symmetry is useful and comes **after** the FOC.

A synergistic relationship

- A similar example from Osborne's textbook
- Two players choosing two positive numbers s_1, s_2
- Payoffs: $v_1\left(s_1,s_2\right)=s_1\left(c+s_2-s_1\right); v_2\left(s_1,s_2\right)=s_2\left(c+s_1-s_2\right)$ where $c>0$
- The best response for i is $s_i^*\left(s_j\right)=\frac{c+s_j}{2}$ for $i,j\in\{1,2\}$ and $i\neq j$
- N.E.: $s_1^*=s_2^*=c$

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Bertrand Competition (Price-setting firms)

- what if firms are setting prices instead of quantities?
- consumers buy from the lowest-price firm

Details:

- Two firms producing at constant unit cost c: $C(q_i) = cq_i$
- Demand:

$$q_i(p_i, p_j) = \begin{cases} a - p_i & \text{if } p_i < p_j \\ \frac{a-p_i}{2} & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

Bertrand Competition (Price-setting firms)

The game

- Players: 2 firms
- Strategies: setting non-negative prices
- Payoffs:

$$\pi_i\left(p_i, p_j\right) = \begin{cases} \left(p_i - c\right)\left(a - p_i\right) & \text{if } p_i < p_j \\ \frac{1}{2}\left(p_i - c\right)\left(a - p_i\right) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

Bertrand Competition (Price-setting firms)

What are the best responses?

- Case 1: If p_j is high I face $q_i\left(p_i, p_j\right) = a - p_i$
- Solving $\max_{p_i} \pi_i\left(p_i, p_j\right)$ I obtain:

$$p_i^M = \frac{a + c}{2}$$

- As long as $p_j > p_i^M$ I choose p_i^M.

Bertrand Competition (Price-setting firms)

- **Case 2**: If p_j is low, namely, $p_j \leq c$ I face $q_i\left(p_i, p_j\right) = 0$.
- I am outside of the market.
- Any p_i is a (trivial) best response.
- **Case 3**: If $c < p_j \leq p_i^M$ it is optimal for me to play just below p_j in order to leave the other firm outside of the market.
- In that case I make $\left(p_i^* - c\right)\left(a - p_i^*\right) = \left(p_j - \varepsilon - c\right)\left(a - p_j - \varepsilon\right)$ with ε a small positive number.
- The other firm will do the same...
- The unique N.E. is $p_i^* = p_j^* = c$.

Bertrand Competition (Price-setting firms)

- Notice the difference in the conclusions of the two models.
- Notice the relevant assumptions e.g. in Bertrand competition every firm is assumed to have the capacity to serve all demand...

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Hotelling: The Setup

- classic location model
- Two players, firm 1 and 2, are located each in opposite extremes of a street of length 1. Firm 1 is located in $x = 0$ and firm 2 in $x = 1$.
- Both firms have cost of production c.
- There is a continuum of costumers uniformly distributed along the street.
- A consumer located in position x incurs a transportation cost tx if buying from firm 1 and $t(1 - x)$ if buying from firm 2.

Hotelling: The Setup

- We assume that consumers buy from the firm that provides the lowest aggregate price: the price plus the transportation cost.
- For given p_1; p_2, the demand that firm 1 gets is given by all customers such that $p_1 + tx \leq p_2 + t(1 - x)$. In particular:

$$D_1(p_1, p_2) = \frac{p_2 - p_1 + t}{2t}$$

- Best response for firm 1:

$$\max_{p_1} D_1(p_1, p_2)(p_1 - c)$$

- From this we get: $p_1^*(p_2) = \frac{p_2+t+c}{2}$ and equivalently $p_2^*(p_1) = \frac{p_1+t+c}{2}$
- Hence $p_1^* = p_2^* = t + c$
- location model applied to electoral competition (Downs, 1957).

Nash Equilibrium

- Augustin Cournot (1838)
- Joseph Bertrand (1883)
- Harold Hotelling (1929)
- John Nash (1950)
- Nash is the one who formalized this way of reasoning as an equilibrium concept that can be used for different social interactions

Nash Equilibrium

- Does a Nash Equilibrium always exist?
- Example: Matching Pennies
- The coins' faces match: Player 1 wins. Otherwise Player 2 wins.

v_1 / v_2	Heads	Tails
Heads	$1, -1$	$-1, 1$
Tails	$-1, 1$	$1, -1$

- Is it always unique? Battle of the sexes.

Nash Equilibrium

- Another example of multiple N.E.
- Hawk-Dove game
- Two players choose between "Aggressive" and "Passive"

v_1 / v_2	Aggressive	Passive
Aggressive	$0, 0$	$4, 2$
Passive	$2, 4$	$3, 3$

Nash Equilibrium

- Thomas Schelling (1921-2016). Nobel prize in 2015.
- Multiple equilibria as a prevalent fact of life not to be ignored, but to be appreciated and understood.
- Sometimes we have a good reason to believe that one of the multiple equilibria is a **focal equilibrium** that the players will coordinate on.

v_1 / v_2	Concert	Home
Concert	$2, 2$	$0, 0$
Home	$0, 0$	$1, 1$

Nash Equilibrium

- Stag-Hunt game

v_1 / v_2	Stag	Hare
Stag	$5, 5$	$0, 3$
Hare	$3, 0$	$3, 3$

- Is multiplicity of equilibria a weakness of Nash equilibrium as a solution concept?
- Or is it a way to understand the heterogeneity in human societies/interactions?

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Revisit the tragedy of the commons

- There is a pasture shared by n local herders, so $I = \{1, 2, ...n\}$
- There is a total stock of grass equal to K, which is the common resource.
- Each herder chooses his consumption of grass $k_i \geq 0$ to feed his cattle. Hence, the remaining grass is $\hat{k} = K - \sum k_i$.
- Herders get benefits not only from consumption of grass, but also from what is left available after individual use. As a particular case, we assume that

$$g_i\left(k_i, k_{-i}\right) = \ln k_i + \ln \hat{k}$$

Revisit the tragedy of the commons

- Best response of herder i:

$$\max_{k_i} \ln k_i + \ln \hat{k}$$

$$\begin{aligned} \frac{dg}{dk_i} &= 0 \\ \frac{1}{k_i} + \frac{1}{\hat{k}}\frac{d\hat{k}}{dk_i} &= 0 \\ \frac{1}{k_i} - \frac{1}{\hat{k}} &= 0 \end{aligned}$$

- In a symmetric equilibrium we have: $\hat{k} = K - nk_i$ which results in $k_i^* = \frac{K}{n+1}$.

Revisit the tragedy of the commons

- A social planner would instead choose

$$\max_{k_1} \ln k_1 + \ln k_2 + ... + \ln k_n + n \ln \hat{k}$$

$$\begin{aligned} \frac{1}{k_1} + \frac{n}{\hat{k}}\frac{d\hat{k}}{dk_1} &= 0 \\ \frac{1}{k_i} - \frac{n}{\hat{k}} &= 0 \end{aligned}$$

- In a symmetric equilibrium we have: $\hat{k} = K - nk_i$ which results in $k_i^{**} = \frac{K}{2n} < k_i^*$.
- You can check that the herder's payoff is larger in the latter case than in the former.
- The N.E. is Pareto dominated.

Summary

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The end

dimitrios.zormpas@cyu.fr