GAME THEORY

Practice exercises - games of complete information

Problem 1:

Consider a static game between two players as described by the payoff matrix below:

		Player 2		
		Left	Center	Right
Player 1	Up	1, 4	1, 2	3, 3
	Middle	4, 4	0, 2	1, 2
	Down	5, 0	2, 1	0, 0

a. What are the actions/strategies sets for the two players?

The set of actions/strategies for player 1 is $S_A = \{Up, Middle, Down\}$ and for player 2 is $S_B = \{Left, Center, Right\}$.

Note that in static games actions and strategies are the same thing!

b. Is there any strictly dominated strategy for player 1? Is there any strictly dominated strategy for player 2? Explain your answer.

Let us fix the choice of player 2 to "Left". Then, for player 1 we get

$$U_1(Down, Left) = 5 > U_1(Middle, Left) = 4 > U_1(Up, Left) = 1$$

that is, "Down" is player 1's best response to player 2's action "Left."

Let us fix the choice of player 2 to "Center". Then, for player 1 we get

$$U_1(Down, Center) = 2 > U_1(Up, Center) = 1 > U_1(Middle, Center) = 0$$

that is, "Down" is player 1's best response to player 2's action "Center."

Let us fix the choice of player 2 to "Right". Then, for player 1 we get

$$U_1(Up, Right) = 3 > U_1(Middle, Right) = 1 > U_1(Down, Right) = 0$$

that is, "Up" is player 1's best response to player 2's action "Right."

We observe that player 1 sometimes chooses "Down" (i.e., when player 2 chooses either "Left" or "Center") and sometimes "Up" (i.e., when player 2 chooses "Right"). Hence there is no strictly dominant strategy for player 1. However, we also observe that player 1 never chooses "Middle," hence this is a strictly dominated strategy for player 1.

Let us fix the choice of player 1 to "Up". Then, for player 2 we get

$$U_2(Up, Left) = 4 > U_2(Up, Right) = 3 > U_2(Up, Center) = 2$$

that is, "Left" is player 2's best response to player 1's action "Up."

Let us fix the choice of player 1 to "Middle". Then, for player 2 we get

$$U_2(Middle, Left) = 4 > U_2(Middle, Right) = U_2(Middle, Center) = 2$$

that is, "Left" is player 2's best response to player 1's action "Middle."

Let us fix the choice of player 1 to "Down". Then, for player 2 we get

$$U_2(Down, Center) = 1 > U_2(Down, Right) = U_2(Down, Left) = 0$$

that is, "Center" is player 2's best response to player 1's action "Down."

We observe that player 2 sometimes chooses "Left" (i.e., when player 1 chooses either "Up" or "Middle") and sometimes "Center" (i.e., when player 1 chooses "Down"). Hence there is no strictly dominant strategy for player 2. However, we also observe that player 2 never chooses "Right," hence this is a strictly dominated strategy for player 2.

c. What is (are) the Nash Equilibrium (Equilibria) in pure strategies? Explain your answer.

Here, we are going to find the Nash Equilibrium using the best responses approach. First, we consider player 1. We fix the strategy of player 2 to "Left." Then it is best for player 1 to choose "Down", hence

		Player 2		
		Left	[redacted]	[redacted]
Player 1	Up	1, 4	[redacted]	[redacted]
	Middle	4, 4	[redacted]	[redacted]
	Down	$\underline{5}$, 0	[redacted]	[redacted]

Then we fix the strategy of player 2 to "Center." Then it is best for player 1 to choose "Down", hence

		Player 2		
		[redacted]	Center	[redacted]
Player 1	Up	[redacted]	1, 2	[redacted]
	Middle	[redacted]	0, 2	[redacted]
	Down	[redacted]	$\underline{2}$, 1	[redacted]

Finally, we fix the strategy of player 2 to "Right." Then it is best for player 1 to choose "Up", hence

		Player 2		
		[redacted]	[redacted]	Right
Player 1	Up	[redacted]	[redacted]	$\underline{3}$, 3
	Middle	[redacted]	[redacted]	1, 2
	Down	[redacted]	[redacted]	0, 0

Now, we consider player 2. We fix the strategy of player 1 to "Up." Then it is best for player 2 to choose "Left", hence

		Player 2		
		Left	Center	Right
Player 1	Up	1, $\underline{4}$	1, 2	3, 3
	[redacted]	[redacted]	[redacted]	[redacted]

Then, we fix the strategy of player 1 to "Middle." Then it is best for player 2 to choose "Left", hence

		Player 2		
		Left	Center	Right
Player 1	[redacted]	[redacted]	[redacted]	[redacted]
	Middle	4, $\underline{\underline{4}}$	0, 2	1, 2
	[redacted]	[redacted]	[redacted]	[redacted]

Finally, we fix the strategy of player 1 to "Down." Then it is best for player 2 to choose "Center", hence

		Player 2		
		Left	Center	Right
Player 1	[redacted]	[redacted]	[redacted]	[redacted]
	Down	5, 0	2, $\underline{\underline{1}}$	0, 0

Putting everything together, the payoff matrix becomes.

		Player 2		
		Left	Center	Right
Player 1	Up	1, $\underline{\underline{4}}$	1, 2	$\underline{\underline{3}}$, 3
	Middle	4, $\underline{\underline{4}}$	0, 2	1, 2
	Down	$\underline{\underline{5}}$, 0	$\underline{\underline{2}}$, $\underline{\underline{1}}$	0, 0

Note that only in the cell (Down, Center) both payoffs are underlined. Hence, the Nash Equilibrium is {"Down", "Center"}.

d. Is there a coordination failure in this game? Explain your answer.

There is a coordination failure in this game. We will show that the Nash equilibrium NE = {"Down", "Center"} is not Pareto efficient. Remember that…

> ***"… an outcome is not Pareto Efficient if there is an alternative outcome in which at least one player is better off without anyone getting worse off."***

Note that the outcome resulting when players 1 and 2 choose "Middle" and "Left", correspondingly, yields 4 for each one of them. Hence both are better off when moving from {"Down", "Center"} to {"Middle", "Left"}.

Problem 2: ***(Games with Positive Externalities)***

Two neighboring countries, $i = 1, 2$, simultaneously choose how many resources (in hours) to spend in recycling activities, r_i. The average benefit (π_i) for every dollar spent on recycling is

$$\pi_i(r_i, r_j) = 10 - r_i + \frac{r_j}{2}$$

and the (opportunity) cost per hour of recycling activity for each country is 4. Country ***i***'s average benefit is increasing in the resources that neighboring country ***j*** spends on recycling due to positive external effects on other countries.

a. Find each country's best-response function, and compute the Nash Equilibrium (r_1^*, r_2^*)

Since the gains of recycling are given by $(\pi_i \cdot r_i)$, and the costs of the activity are $(4r_i)$, country 1's maximization problem consists of selecting the amount of hours devoted to recycling r_1 that solves:

$$\max_{r_1} \left(10 - r_1 + \frac{r_2}{2}\right) r_1 - 4r_1$$

Taking the first-order condition with respect to r_1

$$10 - 2r_1 + \frac{r_2}{2} - 4 = 0$$

Rearranging and solving for r_1 yields country 1's best-response function (BRF_1):

$$r_1(r_2) = 3 + \frac{r_2}{4}$$

Inserting best-response function $r_2(r_1)$ into $r_1(r_2)$ yields

$$r_1 = 3 + \frac{3 + \frac{r_1}{4}}{4},$$

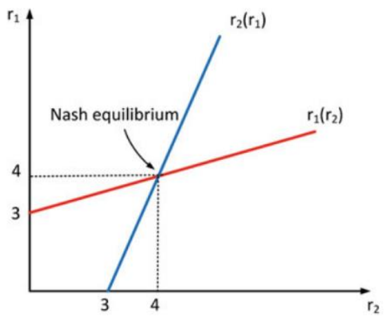
and, rearranging, we obtain an equilibrium level of recycling of $r_1{}^* = 4$ for country 1. Hence, country 2's equilibrium recycling level is

$$r_2^* = 3 + \frac{4}{4} = 4$$

Hence, the psNE is given by:

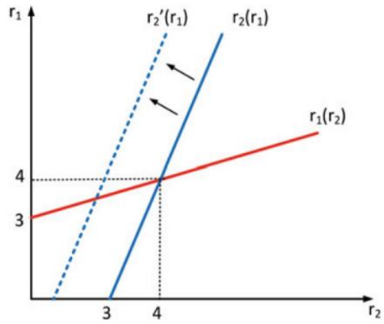
$$psNE = \left\{r_1^* = 4,\ r_2^* = 4\right\}$$

b. Graph the best-response functions and indicate the pure strategy Nash Equilibrium on the graph.



Both best response functions originate at 3 and increase with a positive slope of 1/4, as depicted in the diagram. Intuitively, countries' strategies are strategic complements, since an increase in r_2 induces Country 1 to strategically increase its own level of recycling, r_1, by 1/4.

c. On your previous figure, show how the equilibrium would change if the intercept of one of the countries' average benefit functions fell from 10 to some smaller number.



A reduction in the benefits from recycling produces a fall in the intercept of one of the countries' average benefit function, for example in Country 2. This change is indicated in the diagram by the leftward shift (following the arrow) in Country 2's best response function. In the new Nash Equilibrium, Country 2 recycles a lot less while Country 1 recycles a little less.

<u>Problem 2'</u>: *(Modifying the actions set in the above problem)*

Two neighboring countries, $\boldsymbol{i = 1,2}$, simultaneously choose how many resources (in hours) to spend in recycling activities, $\boldsymbol{r_i}$. However, country 2 is facing a capacity constraint: $\boldsymbol{r_2}$ cannot exceed a certain maximum, say $\boldsymbol{r_{2MAX} = 3,4}$. The average benefit ($\boldsymbol{\pi_i}$) for every dollar spent on recycling is

$$\boldsymbol{\pi_i(r_i, r_j) = 10 - r_i + \frac{r_j}{2}}$$

and the (opportunity) cost per hour of recycling activity for each country is 4. Country $\boldsymbol{i}$'s average benefit is increasing in the resources that neighboring country $\boldsymbol{j}$ spends on his recycling because a clean environment produces positive external effects on other countries.

a. Find each country's best-response function, and compute the Nash Equilibrium $(\boldsymbol{r_1^*, r_2^*})$

Similarly to Problem 2, the best response function of country 1 is given by

$$\boldsymbol{r_1 = 3 + \frac{r_2}{4}} \qquad (1)$$

However, for country 2 we must be careful! Country 2's best response function will be indeed (as in Problem 2)

$$\boldsymbol{r_2 = 3 + \frac{r_1}{4}}$$

only if the outcome of this function is less or equal to $\boldsymbol{r_{2MAX} = 3,4}$, i.e., if

$$\boldsymbol{r_2 = 3 + \frac{r_1}{4} \leq r_{2MAX} = 3,4 \Rightarrow\ 3 + \frac{r_1}{4} \leq 3,4 \Rightarrow \ldots\ r_1 \leq 1,6}$$

In case that $\boldsymbol{r_1 \leq 1}$, country 2 can only choose $\boldsymbol{r_2 = r_{2MAX} = 3,4}$. Therefore, in summary, country 2's best response function is

$$\boldsymbol{r_2 = \begin{cases} 3 + \frac{r_1}{4}\ , & \text{if } r_1 \leq 1,6 \\ 3,4\ \ , & \text{otherwise} \end{cases}} \qquad (2)$$

Solving equations (1)-(2) as a system requires to consider two cases. First, under the assumption that $\boldsymbol{r_1 \leq 1,6}$ we solve the system

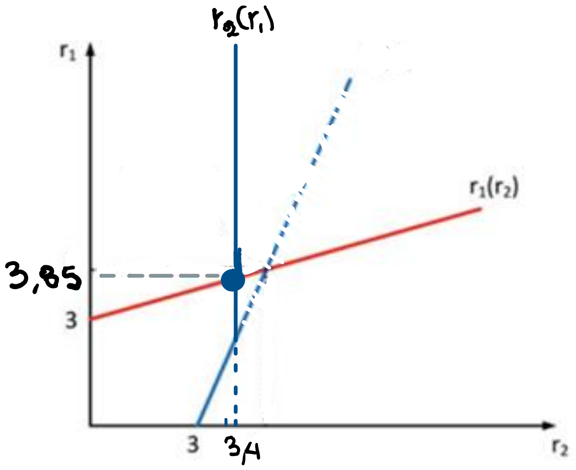
$$\begin{cases} r_1 = 3 + \frac{r_2}{4} \\ \\ r_2 = 3 + \frac{r_1}{4} \end{cases}$$

However, as seen in Problem 2, the solution of this system is $r_1^* = r_2^* = 4$ that violates the condition $r_1 \leq 1,6$. This solution cannot be accepted! Secondly, under the assumption that $r_1 > 1,6$ we solve the system

$$\begin{cases} r_1 = 3 + \frac{r_2}{4} \\ \\ r_2 = 3,4 \end{cases}$$

You can easily verify that the solution of this system is $r_1^* = 3,85$ and $r_2^* = 3,4$. Note that this solution is accepted as it satisfies the condition that $r_1 > 1,6$.

b. Graph the best-response functions and indicate the pure strategy Nash Equilibrium on the graph.



<u>Problem 2'':</u> ***(Positive or negative externalities?)***

Two neighboring countries, $i = 1,2$, simultaneously choose the level of recycling activities, r_i. The average benefit (π_i) on recycling is

$$\pi_i(r_i, r_j) = 10 - r_i + \frac{r_j}{2}$$

Country i's average benefit is increasing in the resources that neighboring country j spends on his recycling because a clean environment produces positive external effects on other countries. The recycling cost depends on the sum of recycling activities of the two countries and is it described by

$$C = \alpha(r_1 + r_2)^2$$

where $a \geq 0$.

a. Formally define the game.

There are two players, countries 1 and 2 → $N = \{1,2\}$

The set of actions/strategies are $r_i \in [0, \infty)$ for both countries.

The payoffs are given by

$$U_i = \pi_i(r_i, r_j) \times r_i - C \Rightarrow$$
$$U_i = \left(10 - r_i + \frac{r_j}{2}\right) r_i - \alpha(r_i + r_j)^2 \Rightarrow$$

b. Find each country's best-response function.

To find the best-response function of country i we maximize the payoff with respect to r_i →

$$\frac{\partial U_i}{\partial r_i} = 0 \Rightarrow\ 10 - 2r_i + \left(\frac{1}{2}\right) r_j - 2ar_i - 2ar_j = 0 \Rightarrow r_i = \frac{10}{2+2a} - \frac{2a - 1/2}{2+2a} r_j$$

Note that depending on the value of a, the coefficient of r_j might be positive or negative, i.e., there might be strategic complementarities or strategic substitutabilities, respectively. Specifically, that coefficient will be negative (positive) if $a > 1/4$ $(a < 1/4)$.

c. Compute the Nash equilibrium of this game.

Solving the 2x2 system of best responses

$$\begin{cases} r_1 = \dfrac{10}{2+2a} - \dfrac{2a - 1/2}{2+2a} r_2 \\ \\ r_2 = \dfrac{10}{2+2a} - \dfrac{2a - 1/2}{2+2a} r_1 \end{cases}$$

yields

$$r_1^* = r_2^* = \frac{20}{3+8a}$$

d. How does the Nash Equilibrium change with changes in a?

Taking the derivative of the optimal level of recycling yields

$$\frac{\partial r_i^*}{\partial r_i} = -\frac{160}{(3+8a)^2} < 0$$

Implying that the solution decreases as a increases.

<u>Problem 3:</u>

A luxury car manufacturer is producing "handmade" cars. The production of every single car requires assembling different parts, each prepared by a highly skilled worker. A car can only be produced if all necessary parts are available (i.e., car parts are perfect complements in production). The production level of the different parts depends on the effort a worker is putting on the job. Every worker $i \in \{1, 2, \dots, 10\}$ chooses an effort level $e_i \in [0, 1]$. The payoff to worker i is given by

$$\pi_i(e_1, \dots, e_{10})\ =\ 5\, min\{e_1, \dots, e_{10}\} - 2e_i$$

a. Find all the Nash equilibria and prove that they are indeed Nash equilibria.

<u>Claim 1:</u> There is no Nash Equilibrium where any two workers $i \neq j$ choose different effort level.

<u>Proof:</u> Let there be an equilibrium $(e_1^*, e_2^*, \dots, e_n^*)$ with $e_i^* < e_j^*$ for two workers i and j. Then by construction $min(e_1^*, e_2^*, \dots, e_n^*) = e_i^*$. Worker j's payoff is then

$$\pi_j^*(e_1^*, e_2^*, \dots, e_j^*, \dots, e_n^*) = 5\, e_i^* - 2e_j^*$$

However, should worker j reduces her effort by an amount $\varepsilon \to 0$, i.e., $e_j^{**} = e_j^* - \varepsilon$, her payoff becomes

$$\pi_j^{**}(e_1^*, e_2^*, \dots, e_j^{**}, \dots, e_n^*) = 5\, e_i^* - 2e_j^* + \varepsilon$$

Since $\pi_j^{**}(e_1^*, e_2^*, \dots, e_j^{**}, \dots, e_n^*) - \pi_j^*(e_1^*, e_2^*, \dots, e_j^*, \dots, e_n^*) = \varepsilon > 0$ worker j has an incentive to deviate from the suggested equilibrium strategy. Hence, the strategy profile $(e_1^*, e_2^*, \dots, e_n^*)$ cannot be a Nash Equilibrium.

<u>Claim 2:</u> Any strategy profile where all workers choose the same effort level is a Nash Equilibrium.

<u>Proof:</u> Let there be an equilibrium $(e_1^*, e_2^*, \dots, e_n^*)$ with $e_i^* = e_j^* = e$ for all workers $i \neq j$. Then by construction $e = min(e_1^*, e_2^*, \dots, e_n^*)$. Worker i's payoff is then

$$\pi_i^*(e_1^*, e_2^*, \dots, e_i^*, \dots, e_n^*) = 5\, e - 2e = 3e$$

- **should worker i reduces her effort by an amount $\varepsilon \to 0$, i.e., $e_i^{**} = e - \varepsilon$, we get $min(e_1^*, e_2^*, \dots, e_n^*) = e - \varepsilon$, and her payoff becomes**
$$\pi_i^{**}(e_1^*, e_2^*, \dots, e_i^{**}, \dots, e_n^*) = 5\,(e - \varepsilon) - 2(e - \varepsilon) = 3e - 3\varepsilon$$

Since $\pi_i^{**}(e_1^*, e_2^*, \dots, e_i^{**}, \dots, e_n^*) - \pi_j^*(e_1^*, e_2^*, \dots, e_i^*, \dots, e_n^*) = -3\varepsilon < 0$ worker i has no incentive to deviate from the suggested equilibrium strategy by decreasing her effort.

- **should worker i increases her effort by an amount $\varepsilon \to 0$, i.e., $e_i^{\wedge} = e + \varepsilon$, we get $min(e_1^*, e_2^*, \dots, e_n^*) = e$, and her payoff becomes**
$$\pi_i^{\wedge}(e_1^*, e_2^*, \dots, e_i^{\wedge}, \dots, e_n^*) = 5\, e - 2(e + \varepsilon) = 3e - 2\varepsilon$$

Since $\pi_i^{**}(e_1^*, e_2^*, \dots, e_i^{**}, \dots, e_n^*) - \pi_j^*(e_1^*, e_2^*, \dots, e_i^*, \dots, e_n^*) = -2\varepsilon < 0$ worker i has no incentive to deviate from the suggested equilibrium strategy by increasing her effort.

Hence, no worker has an incentive to unilaterally change her effort level.

b. Are any of the Nash equilibria Pareto efficient? Justify your answer.

Any symmetric equilibrium with $e < 1$ is Pareto inefficient, because all the players would be better off if they collectively switched to $(1,\ 1, \dots,\ 1)$. On the other hand, the symmetric equilibrium $(1,\ 1, \dots,\ 1)$ is Pareto efficient. The proof of the above statement is obvious and left to the student.

c. Suggest a mechanism to facilitate cooperation so that the Pareto Efficient Nash Equilibrium becomes the only equilibrium.

One can think of different mechanisms here. For example,

- **any mechanism that facilitates communication between the workers. In practical terms, a labor union can act as the mean of communication between the workers. Of course, we should note here that allowing for communication before the "game is played" changes the nature of the game (i.e., it will not be a static non-cooperative game anymore)**
- **a mechanism that alters the payoffs so that the Pareto efficient outcome becomes the only Nash Equilibrium. For example, the owner of the company**

has an incentive to get the highest effort level from every worker since this ensures the maximum production level (and maximum profit if there is a constant profit margin per unit produced). Hence the owner can offer wages that either reward maximum effort or penalize non-maximum effort. Such a wage can be

$$w = \begin{cases} 4min\{e_1, \dots, e_n\}, & \text{if } \exists e_i < 1 \text{ for at least one } i \in N \\ 5, & \text{if } e_i = 1 \text{ for all } i \in N \end{cases}$$

Problem 4:

Let there be two players, Anna and Beth, each having to put in an opaque jar either €0, €400 or €700 of her own money. A third person, Charles, collects the money and counts the total amount, say X. Then Charles adds on X an amount of his own money according to the following rule:

- 0% of the total if $X \leq €400$
- 50% of the total if $€400 < X \leq €1100$
- 100% of the total if $X > €1100$

Finally, Charles splits the overall amount (i.e., the total AND his own contribution) equally between Anna and Beth. Anna and Beth care only about their net benefits.

a. Fully describe the game and construct the payoff matrix.

The players of this game are Anna and Beth as they must choose among available strategies (€0, €400 or €700) and whose payoffs depend on what the final outcome will be. Note that Charles is not a player: his actions are predetermined (not a choice) and there is no payoff involved for Charles in this set up. To derive the payoff matrix, we can first use the information provided to identify the amount of money distributed to each player. For example, following the rules Charles has set, should both players choose €0 the sum is zero and Charles do not add any amount. Each player will then receive half of €0, i.e., zero! If, for example, Anna chooses €400 while Beth chooses €700 the sum is €1100 and Charles will add 50% of €1100 which equals €550. The total amount, after Charles' contribution is €1650 and each player will then receive half of €1650, i.e., €825! Working similarly, we get the following matrix depicting the distributed amounts:

		Beth		
		€0	€400	€700
Anna	€0	0 , 0	200 , 200	525 , 525
	€400	200 , 200	600 , 600	825 , 825
	€700	525 , 525	825, 825	1400 , 1400

Using the above matrix with the distributed amounts of money we can generate the the 2-player 3x3 matrix below that fully describes the game, by subtracting from each player the amounts they paid at the beginning of the game. For example, when Anna chooses €400 and Beth chooses €700, Charles adds another €550 and each player receives half of €1650, i.e., €825! However, Anna's net amount is €825-€400=€425 while Beth's net amount is €825-€700=€125. Following similar logic yields:

		Beth		
		€0	€400	€700
Anna	€0	0 , 0	200 , -200	525 , -175
	€400	-200 , 200	200 , 200	425 , 125
	€700	-175 , 525	125 , 425	700 , 700

b. Is there a strictly dominant action for either player? Is there a strictly dominated action for either player? Fully explain your answer.

No player has any strictly dominant or strictly dominated action. A simple way to confirm this claim is to verify that each player (for example, Anna) uses all of her available strategies as a best response to some choice of her rival: If Beth chooses €0, Anna's best response is to choose €0; If Beth chooses €400, Anna's best response is to choose either €0 or €400; If Beth chooses €700, Anna's best response is to choose €700. Note that due to the symmetry of the game, a similar argument is true for Beth's best responses.

c. Find the Nash Equilibrium (or equilibria) of this game. Is there a problem with coordination failure?

In the payoff matrix of the game we highlight best responses (as we have seen in class) resulting in

		Beth		
		€0	€400	€700
Anna	€0	0 , 0	200 , -200	525 , -175
	€400	-200 , 200	200 , 200	425 , 125
	€700	-175 , 525	125 , 425	700 , 700

We can then confirm that there are three Nash Equilibria in this game, namely

$$\boldsymbol{NE1 = \{€0\,, €0\}}$$
$$\boldsymbol{NE2 = \{€400\,, €400\}}$$
$$\boldsymbol{NE3 = \{€700\,, €700\}}$$

It is also straightforward to confirm that there is a unique Pareto Efficient outcome in this game, namely the outcome resulting from both players choosing €700. Therefore, we have a situation of multiple Nash Equilibria where one of them is Pareto Efficient: this is a case of coordination failure that resembles the generic case of "Pure Coordination Games".

Problem 5:** (*exercise 35.2 in Osborne and Rubinstein*)

Two investors are involved in a competition with a prize of \$1. Each investor can spend any amount in the interval [0,1]. The winner is the investor who spends the most; in the event of a tie each investor receives \$0.50. Formulate this situation as a strategic game and find its mixed strategy Nash equilibria.

The set of actions of each player i is $\boldsymbol{A_i = [0, 1]}$. The payoff function of player i is

$$u_i(a_1, a_2) = \begin{cases} -a_i & \text{if } a_i < a_j \\ \frac{1}{2} - a_i & \text{if } a_i = a_j \\ 1 - a_i & \text{if } a_i > a_j, \end{cases}$$

where $j \in \{1, 2\} \setminus \{i\}$.

We can represent a mixed strategy of a player i in this game by a probability distribution function F_i on the interval $[0,1]$, with the interpretation that $F_i(v)$ is the probability that player i chooses an action in the interval $[0,v]$. Define the *support* of F_i to be the set of points v for which $F_i(v+\epsilon) - F_i(v-\epsilon) > 0$ for all $\epsilon > 0$, and define v to be an *atom* of F_i if $F_i(v) > \lim_{\epsilon \downarrow 0} F_i(v-\epsilon)$. Suppose that (F_1^*, F_2^*) is a mixed strategy Nash equilibrium of the game and let S_i^* be the support of F_i^* for $i = 1, 2$.

Step . $S_1^* = S_2^*$.

Proof. If not then there is an open interval, say (v, w), to which F_i^* assigns positive probability while F_j^* assigns zero probability (for some i, j). But then i can increase his payoff by transferring probability to smaller values within the interval (since this does not affect the probability that he wins or loses, but increases his payoff in both cases).

Step . If v is an atom of F_i^* then it is not an atom of F_j^* and for some $\epsilon > 0$ the set S_j^* contains no point in $(v-\epsilon, v)$.

Proof. If v is an atom of F_i^* then for some $\epsilon > 0$, no action in $(v-\epsilon, v]$ is optimal for player j since by moving any probability mass in F_i^* that is in this interval to either $v+\delta$ for some small $\delta > 0$ (if $v < 1$) or 0 (if $v = 1$), player j increases his payoff.

Step . If $v > 0$ then v is not an atom of F_i^* for $i = 1, 2$.

Proof. If $v > 0$ is an atom of F_i^* then, using Step 2, player i can increase his payoff by transferring the probability attached to the atom to a smaller point in the interval $(v-\epsilon, v)$.

Step . $S_i^* = [0, M]$ for some $M > 0$ for $i = 1, 2$.

Proof. Suppose that $v \notin S_i^*$ and let $w^* = \inf\{w : w \in S_i^* \text{ and } w \geq v\} > v$. By Step 1 we have $w^* \in S_j^*$, and hence, given that w^* is not an atom of F_i^* by Step 3, we require j's payoff at w^* to be no less than his payoff at v. Hence $w^* = v$. By Step 2 at most one distribution has an atom at 0, so $M > 0$.

Step . $S_i^* = [0,1]$ and $F_i^*(v) = v$ for $v \in [0,1]$ and $i = 1, 2$.

Proof. By Steps 2 and 3 each equilibrium distribution is atomless, except possibly at 0, where at most one distribution, say F_i^*, has an atom. The payoff of j at $v > 0$ is $F_i^*(v) - v$, where $i \neq j$. Thus the constancy of i's payoff on $[0, M]$ and $F_j^*(0) = 0$ requires that $F_j^*(v) = v$, which implies that $M = 1$. The constancy of j's payoff then implies that $F_i^*(v) = v$.

We conclude that the game has a unique mixed strategy equilibrium, in which each player's probability distribution is uniform on $[0,1]$.

Problem 6*: Litigation and Posner's nuisance rule. *Posner, R. 1997. Economic analysis of Law, 9th edition.*

ChemCo operates a chemical plant, which is located on the banks of a river. Downstream from the chemical plant is a group of fisheries. The ChemCo plant emits byproducts that pollute the river, causing harm to the fisheries. The profit ChemCo obtains from operating the chemical plant is $\$\boldsymbol{\Pi} > \mathbf{0}$. The harm inflicted on the fisheries due to water pollution is equal to $\$\boldsymbol{L} > \mathbf{0}$ of lost profit [without pollution the fisheries' profit is $\$\boldsymbol{A}$, while with pollution it is $\$(\boldsymbol{A} - \boldsymbol{L})$]. Suppose that the fisheries collectively sue the ChemCo Corporation. It is easily verified in court that ChemCo's plant pollutes the river. However, the values of $\boldsymbol{\Pi}$ and $\boldsymbol{L}$ cannot be verified by the court, although they are commonly known to the litigants. Suppose that the court requires the ChemCo attorney (Player 1) and the fisheries' attorney (Player 2) to play the following litigation game. Player 1 is asked to announce a number $\boldsymbol{x} \geq \mathbf{0}$, which the court interprets as a claim about the plant's profits. Player 2 is asked to announce a number $y \geq 0$, which the court interprets as the fisheries' claim about their profit loss. The announcements are made simultaneously and independently. Then the court uses Posner's nuisance rule to make its decision. According to the rule, if $y > x$, then ChemCo must shut down its chemical plant. If $x \geq y$, then the court allows ChemCo to operate the plant, but the court also requires ChemCo to pay the fisheries the amount y. Note that the court cannot force the attorneys to tell the truth: in fact, it would not be able to tell whether the lawyers were reporting truthfully. Assume that the attorneys want to maximize the payoff (profits) of their clients.

(a) Represent this situation as a normal-form game by describing the strategy set of each player and the payoff functions.

The strategy sets are $S_1 = S_2 = [0, \infty)$. The payoff functions are as follows:[10]

$$\pi_1(x,y) = \begin{cases} \Pi - y & \text{if } x \geq y \\ 0 & \text{if } y > x \end{cases} \quad \text{and} \quad \pi_2(x,y) = \begin{cases} A - L + y & \text{if } x \geq y \\ A & \text{if } y > x \end{cases}$$

(b) Is it a dominant strategy for the ChemCo attorney to make a truthful announcement (i.e., to choose $x = \Pi$)?

Yes, for player 1 choosing $x = \Pi$ is a weakly dominant strategy.

Proof. Consider an arbitrary y. We must show that $x = \Pi$ gives at least a high a payoff against y as any other x. Three cases are possible.

Case 1: $y < \Pi$. In this case $x = \Pi$ or any other x such that $x \geq y$ yields $\pi_1 = \Pi - y > 0$, while any $x < y$ yields $\pi_1 = 0$.

Case 2: $y = \Pi$. In this case 1's payoff is zero no matter what x he chooses.

Case 3: $y > \Pi$. In this case $x = \Pi$ or any other x such that $x < y$ yields $\pi_1 = 0$, while any $x \geq y$ yields $\pi_1 = \Pi - y < 0$.

(c) Is it a dominant strategy for the fisheries' attorney to make a truthful announcement (i.e., to choose $y = L$)?

No, choosing $y = L$ is not a dominant strategy for Player 2. For example, if $x > L$ then choosing $y = L$ yields $\pi_2 = A$ while choosing a y such that $L < y \leq x$ yields $\pi_2 = A - L + y > A$.

(d) For the case where $\Pi > L$, find all the Nash equilibria of the litigation game.

Suppose that $\Pi > L$. If (x,y) is a Nash equilibrium **with** $x \geq y$ then it must be that $y \leq \Pi$ (otherwise Player 1 could increase its payoff by reducing x below y) and $y \geq L$ (otherwise Player 2 would be better off by increasing y above x).

Thus it must be $L \leq y \leq \Pi$, which is possible, given our assumption.

However, it cannot be that $x > y$, because Player 2 would be getting a higher payoff by increasing y to x.

Thus it must be $x \leq y$, which (together with our hypothesis that $x \geq y$) implies that $x = y$. Thus the following are Nash equilibria:

all the pairs (x,y) with $L \leq y \leq \Pi$ and $x = y$.

Now consider pairs (x,y) **with** $x < y$. Then it cannot be that $y < \Pi$, because Player 1 could increase its payoff by increasing x to y. Thus it must be $y \geq \Pi$ (hence, by our supposition that $\Pi > L$, $y > L$). Furthermore, it must be that $x \leq L$ (otherwise Player 2 could increase its profits by reducing y to (or below) x. Thus

(x,y) with $x < y$ is a Nash equilibrium if and only if $x \leq L$ and $y \geq \Pi$.

(e) For the case where $\Pi < L$, find all the Nash equilibria of the litigation game.

Suppose that $\Pi < L$. For the same reasons given above, an equilibrium with $x \geq y$ requires $L \leq y \leq \Pi$. However, this is not possible given that $\Pi < L$. Hence,

there is no Nash equilibrium (x, y) with $x \geq y$.

Thus we must restrict attention to pairs (x, y) **with** $x < y$. As explained above, it must be that $y \geq \Pi$ and $x \leq L$. Thus,

(x, y) with $x < y$ is a Nash equilibrium if and only if $\Pi \leq y$ and $x \leq L$.

(f) Does the court rule give rise to a Pareto efficient outcome?

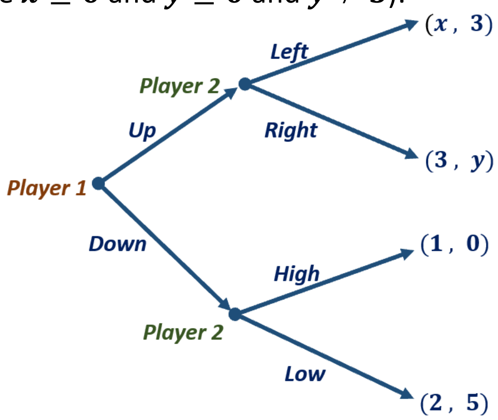
Pareto efficiency requires that the chemical plant be shut down if $\Pi < L$ and that it remain operational if $\Pi > L$.

Now, when $\Pi < L$ all the equilibria have $x < y$ which leads to shut-down, hence a Pareto efficient outcome.

When $\Pi > L$, there are two types of equilibria: one where $x = y$ and the plant remains operational (a Pareto efficient outcome) and the other where $x < y$ in which case the plant shuts down, yielding a Pareto inefficient outcome. □

Problem 7:

Consider the perfect-information extensive form game that is represented by the game tree below (where $x \geq 0$ and $y \geq 0$ and $y \neq 3$):



a. What are the set of strategies for the two players? Also, clearly identify all the subgames.

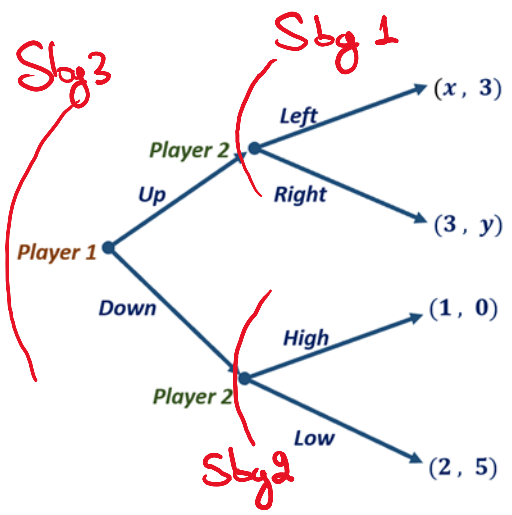
Player 1 has one decision node; thus, his strategies contain only one action. Player 1's strategy set is, therefore,

$$S_1 = \{\text{Up, Down}\}$$

Player 2 has two decision nodes; thus, his strategies contain two actions one for each node. Player 2's strategy set is, therefore,

$$S_2 = \{\text{Left-High, Left-Low, Right-High, Right-Low}\}$$

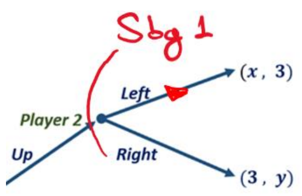
The subgames are shown below: Starting from the end of the game (a) Subgame 1 (Sbg1) consists of player 2 deciding to go Left or Right after player 1 has chosen to go Up, (b) Subgame 2 (Sbg2) consists of player 2 deciding to go High or Low after player 1 has chosen to go Down, and (c) Subgame 3 (Sbg3) consists of player 1 deciding to go Up or Down (given the solutions of subgames 2 and 3).



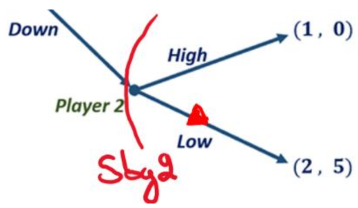
b. Find the Sub-game Perfect Nash Equilibrium (or equilibria) if $\mathbf{y < 3}$.

We proceed solving this game backwards assuming $\mathbf{y < 3}$.

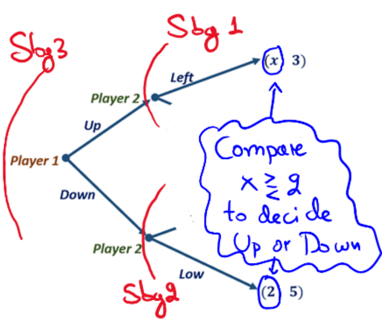
- **In subgame 1 player 2 is the player with the move and she is facing two options. If she chooses to go Left her payoff will be 3 while is she goes Right her payoff will be y. Given our assumption that $\mathbf{y < 3}$, player 2 will choose Left.**



- **In subgame 2 player 2 is the player with the move and she is facing two options. If she chooses to go High her payoff will be 0 while is she goes Low her payoff will be 5. Therefore, she will choose Low.**



- **In subgame 3 player 1 is the player with the move and, given the solutions of subgames 1 and 2, she is facing two options. If she chooses to go Up player 2 in subgame 1 will go Left and player 1 will receive x. If she chooses to go Down player 2 in subgame 1 will go Low and player 1 will receive 2. Hence, player 1 will choose to go Up if $x > 2$, she will prefer to go Down if $x < 2$, and she will be indifferent between her two options if $x = 2$.**



Therefore, the SPNEs are

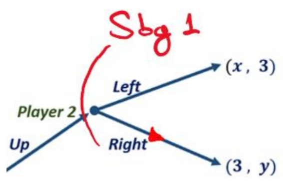
$SPNE_1 = \{\text{Up; Left-Low}\}$, **provided** $x \geq 2$

$SPNE_2 = \{\text{Down; Left-Low}\}$, **provided** $x \leq 2$

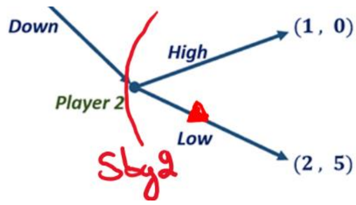
c. Find the Sub-game Perfect Nash Equilibrium (or equilibria) if $\mathbf{y > 3}$.

We proceed solving this game backwards assuming $\mathbf{y > 3}$.

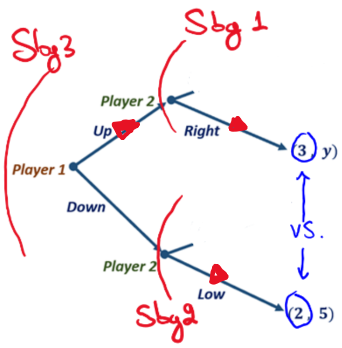
- **In subgame 1 player 2 is the player with the move and she is facing two options. If she chooses to go Left her payoff will be 3 while is she goes Right her payoff will be y. Given our assumption that $\mathbf{y > 3}$, player 2 will choose Right.**



- **In subgame 2 player 2 is the player with the move and she is facing two options. If she chooses to go High her payoff will be 0 while is she goes Low her payoff will be 5. Therefore, she will choose Low.**



- **In subgame 3 player 1 is the player with the move and, given the solutions of subgames 1 and 2, she is facing two options. If she chooses to go Up player 2 in subgame 1 will go Right and player 1 will receive 3. If she chooses to go Down player 2 in subgame 1 will go Low and player 1 will receive 2. Therefore, player 1 will go Up.**

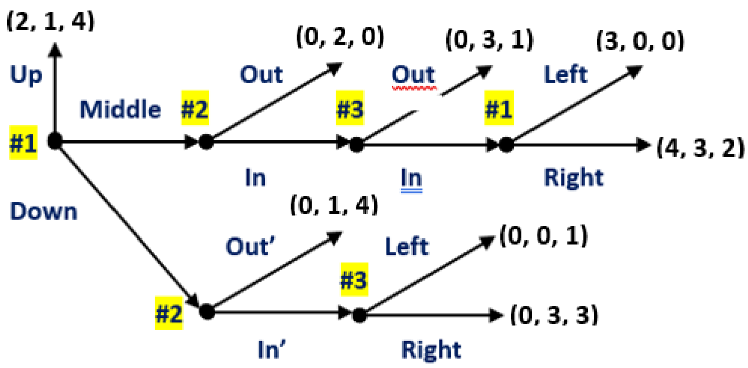


Therefore, the SPNE is

$$SPNE = \{\text{Up; Right-Low}\}$$

<u>Problem 8:</u>

Consider a sequential game between three players described by the game tree below:



a. How many actions should a strategy for player 1, player 2, and player 3, correspondingly, include?

By checking the game tree, we can confirm that each player has two decision nodes. Therefore, any strategy of any player of that game must include two actions.

Although not asked in the question, we can confirm that the strategy sets of the three players are

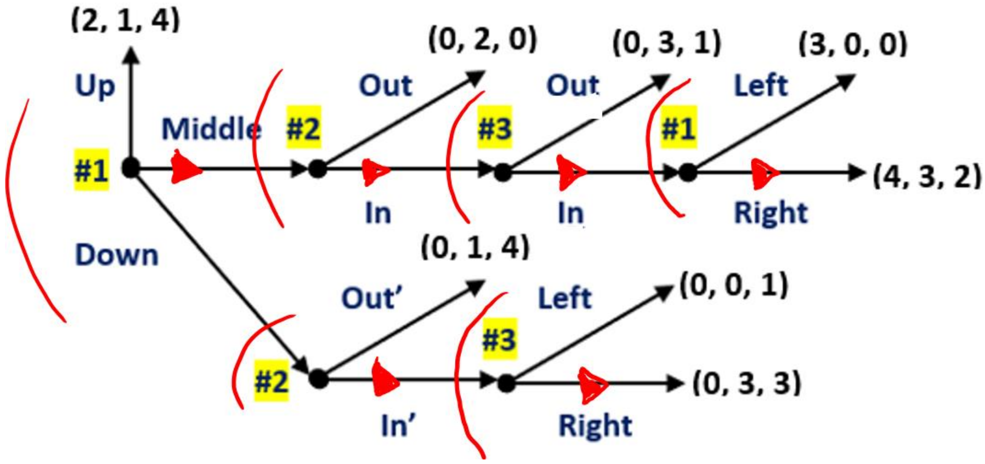
$$S_1 = \{\text{Up-Left, Up-Right, Middle-Left, Middle-Right,Down-Left, Down-Right}\}$$

$$S_2 = \{\text{Out-Out', Out-In', In-Out', In-In'}\}$$

$$S_3 = \{\text{Out-Left, Out-Right, In-Left, In-Right}\}$$

b. What is (are) the Subgame Perfect Nash Equilibrium (Equilibria)?

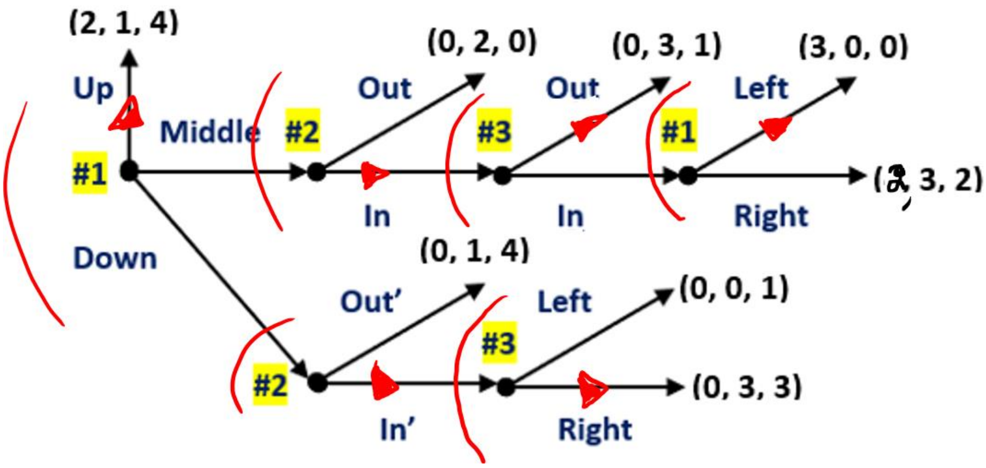
(Only the game-tree is provided here with arrows showing the solution of each subgame; remember that, in any subgame, the player with the move will choose the action that will give her the highest payoff, given the action of players that might follow).



Note that

$$SPNE = \{\text{Middle-Right; In-In'; In-Right}\}$$

c. How would your answer in part (b) change IF the payoffs after player 1 choosing "Right" are **(2, 3, 2)** instead of **(4, 3, 2)**?



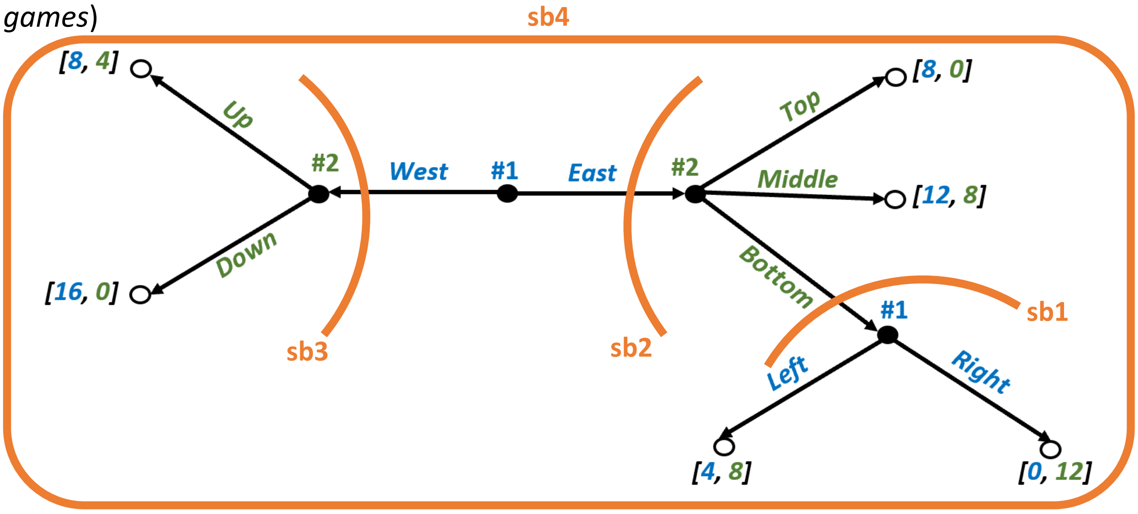
Note that

$$SPNE = \{\text{Up-Left; In-In'; Out-Right}\}$$

Problem 9:

Consider a sequential game between three players described by the game tree below and answer the relevant questions.

(*Note that in the game tree below we have distinguished and numbered the sub-games*)



a. What is the set of actions available to player 1?

{West, East} in the first node and {Left, Right} in the second node that belong to her.

b. What is the set of actions available to player 2?

{Up, Down} in the left node and {Top, Middle, Bottom} in the right node that belong to him.

c. What is the set of strategies available to player 1?

S1={West-Left, West-Right, East-Left, East-Right}

Remember: for the strategies we combine one action on every decision node that belong to a specific player

d. What is the set of strategies available to player 2?

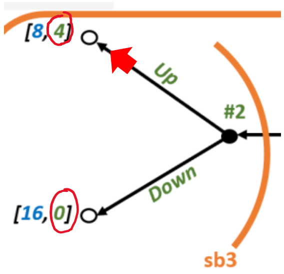
S2={Up-Top, Up-Middle, Up-Bottom, Down-Top, Down-Middle, Down-Bottom}

Remember: for the strategies we combine one action on every decision node that belong to a specific player

e. Fully describe the Sub-Game Perfect Nash Equilibrium (or Equilibria) of this game and justify your answer.

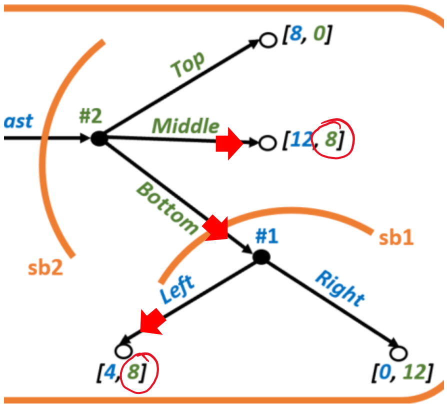
There are two Sub-game Perfect Nash Equilibria in this game. Before we describe them, note that

- **in sub-game 3 below the choice of player 2 is obvious: should player 1 play "West", player 2 will choose "Up" to receive 4 (instead of playing "Down" to receive zero).**

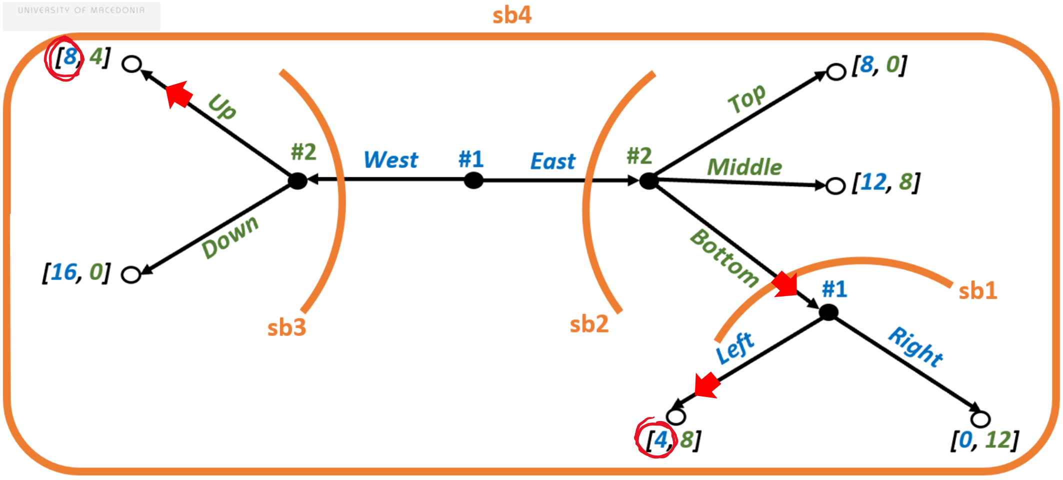


- in sub-game 1 below the choice of player 1 is obvious: should player 1 play "West", player 1 choose "East" and player 2 responds "Bottom", player 1 will choose "Left" to receive 4 (instead of playing "Right" to receive zero).

- in sub-game 2 below, given that choice of player 1 in subgame 1, player 2 is indifferent between "Middle" and "Bottom" since in either case she receives 8 (while, if she plays "Top" she receives zero). <u>What player 2 is assumed to choose in sub-game 2 affects the behavior of player 1 in sub-game 4!</u>

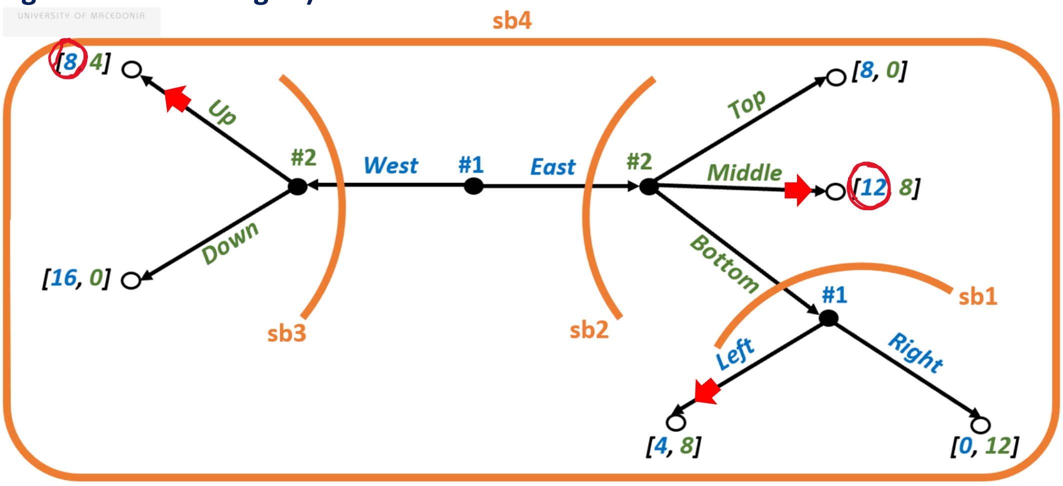


*Let us assume that player 2 in sub-game 2 chooses "Bottom". Then when considering sub-game 4 player 1 sees the following game tree (with arrows showing sub-game Nash strategies):



The payoffs player 1 is facing are 8 if he goes "West" and 4 if he goes "East". Therefore, player 1 will choose "West" in sub-game 4 and a sub-game perfect Nash equilibrium of the game is SPNE1={West-Left, Up-Bottom}

****Let us assume that player 2 in sub-game 2 chooses "Middle". Then when considering sub-game 4 player 1 sees the following game tree (with arrows showing sub-game Nash strategies):**



The payoffs player 1 is facing are 8 if he goes "West" and 12 if he goes "East". Therefore, player 1 will choose "East" in sub-game 4 and a sub-game perfect Nash equilibrium of the game is SPNE2={East-Left, Up-Middle}

Problem 10:

A three-man board, composed of Adam, Brian, and Carol, has been appointed by the City Council to vote for a new Police Chief. There are three candidates, namely X, Y, and Z. During the discussions between the board members, it has become clear to all three of them that their separate opinions are as follows:

- Adam's most preferred choice is candidate X, followed by candidate Y, and Z is the least preferred (we can assume $U_A(X)=2$, $U_A(Y)=1$, $U_A(Z)=0$).
- Brian's most preferred choice is candidate Y, followed by candidate Z, and X is the least preferred (we can assume $U_B(Y)=2$, $U_B(Z)=1$, $U_B(X)=0$)
- Carol's most preferred choice is candidate Z, followed by candidate X, and Y is the least preferred (we can assume $U_C(Z)=2$, $U_C(X)=1$, $U_C(Y)=0$).

Consider the following voting procedure. A member of the board proposes one of the candidates. Then a second member of the board can accept the proposal made by the first member making it a final decision. If the second member disagrees with the proposal of the first member the final decision will be made by the third board member who will have the full power to choose whoever she prefers. However, before they start, Adam has the power to decide who will start, who will respond second, and who will be the third board member to follow. What is the order in which the three board members will play this game and who will be elected as a new Police Chief? Justify your answer.

The orders of play Adam can propose are the ones that can benefit him the most, i.e., the orders of play that will result in candidate X being elected as the new Police Chief. These orders are {Adam, Carol, Brian} and {Carol, Adam, Brian}.

Some easy to prove/confirm facts:

- **There are six potential orders of play, namely {ABC, ACB, BAC, BCA, CAB, CBA}. Because the game is fully symmetric (except from the fact that Adam is the one to choose the order of play) we should expect that each player can have her/his most favored candidate being elected in two out of six potential orders of play.**
- **Whoever plays last has a strictly dominant strategy to choose her/his most preferred candidate: once the last player must decide, she/he has no incentive to choose anything other than her/his best candidate!**
- **Whoever plays last will NEVER (in equilibrium) have the chance to cast her/his vote! (i.e., the most favored candidate of the last player is NEVER elected!). Therefore, Adam will never choose the orders of play BCA and CBA. Why?**
  - **Say the chosen order is BCA. For Adam to have the chance to cast his vote, it must be the case that Carol has rejected the proposal of Brian. In this case X will be elected and the payoffs are 2 for Adam, 1 for Carol and 0 for Brian. But, will Brian ever knowingly make a proposal that Carol will reject IF he knows that this will lead him to a payoff of 0? What are Brian's options? Brian will never propose X because it is his least preferred option and independently of Carol decision (accept or reject) his final payoff will be 0. If, on the other hand, Brian proposes Y then Carol will definitely reject (because Y is Carol's least preferred candidate) and Adam will choose X, so Brian's payoff will be 0. Therefore, Brian will propose Y (his second most preferred candidate) and Carol will gladly accept!**
    - **Note that, based on the discussion above, the player who plays second will see her/his most preferred candidate being elected IF the player who chooses first has the second player's most preferred candidate as her/his second choice. Given this, CAB is an order that will result (in equilibrium) in X being elected!**
  - **Say the chosen order is CBA. For Adam to have the chance to cast his vote, it must be the case that Brian has rejected the Carol's proposal. In this case X will be elected and the payoffs are 2 for Adam, 1 for Carol, and 0 for Brian. But, can Carol do better than a payoff of 1? Carol knows that Brian has no incentive to reject her proposal should it give him a payoff more than 0. That it, Brian will accept Carol's proposal should that be Y or Z. But, in this case, it is on the best interest of Carol to propose Z (her most preferred**

candidate). Brian will accept because he knows that his rejection will lead him a worst outcome!

- **Note that, based on the discussion above, the player who plays first will see her/his most preferred candidate being elected IF the player who chooses second has the first player's most preferred candidate as her/his second choice. Given this, ACB is an order that will result (in equilibrium) in X being elected!**

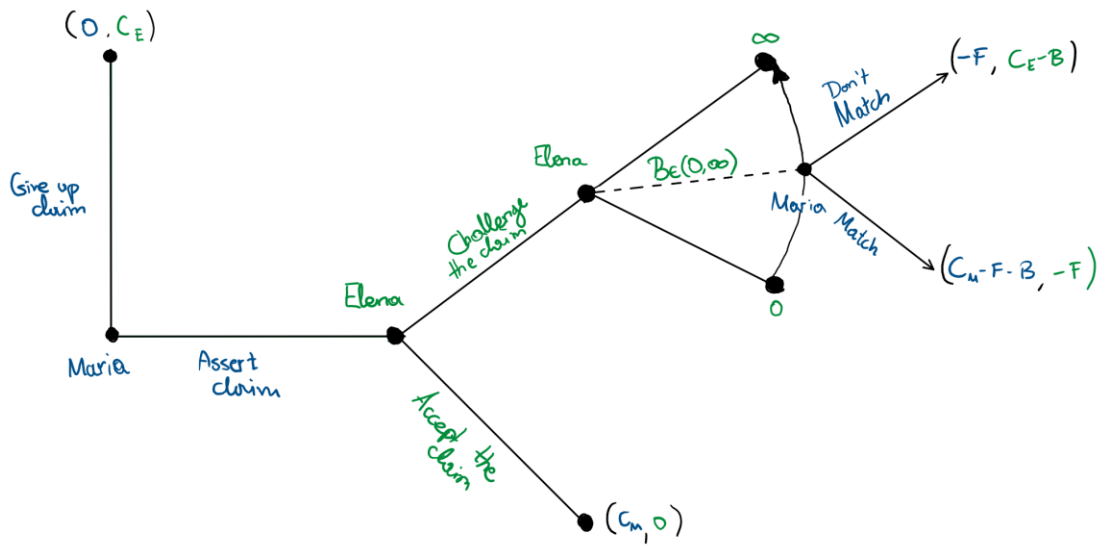
Problem 11: *(Giacomo Bonanno-Game Theory)*

Two different women, Maria and Elena, claim to be the owners of an antique painting of sentimental value. Let C_M denote Maria's and C_E Elena's monetary equivalent of getting the painting. Not getting the painting yields, for both, a monetary equivalent of zero euros. However, none of them possesses evidence of ownership. A judge wants the painting to be returned to its legal owner. He knows that one of the two is indeed the legal owner, but he does not know who that is. He comes with the following idea. A fine $F > 0$ is announced and then the judge asks Maria and Elena to play the following game.

- In the first stage, Maria either gives up her claim to the painting (in which case the game ends with Elena getting the painting and nobody paying the fine) or she asserts her claim, in which case the game proceeds.
- In the second stage, Elena either accepts Maria's claim (in which case the game ends with Maria getting the painting and nobody paying the fine) or challenges her claim. In the latter case, Elena must put in a bid $B > 0$ and the game proceeds.
- In the final stage, Maria can either match Elena's bid (in which case Maria gets the painting by paying $F + B$, and Elena pays the fine F) or chooses not to match (in which case Elena pays B and gets the painting while Maria pays F).

Under what conditions a sub-game perfect Nash equilibrium with the painting returning to its legal owner can be supported?

The game can be represented by the following tree



There are two possibilities: (a) Maria is the owner, or (b) Elena is the owner.

(a) Consider first the case where Maria is the owner.

- Suppose that Maria values the painting more than Elena does, i.e. $C_M > C_E$. At the last stage Maria will choose to match Elena's bid if $C_M > B$ and not to match Elena's bid if $C_M < B$. In the first case Elena's payoff is $-F$, while in the second case it will be $C_E - B$, which is negative since $C_M < B$ and $C_M > C_E$. Therefore, in either case Elena's payoff would be negative. Hence at her decision node Elena will choose to accept Maria's claim getting a payoff of zero. Anticipating this, Maria will assert her claim at the first decision node. Therefore, at the backward-induction solution the painting goes to Maria, the legal owner. The payoffs are C_M for Maria and 0 for Elena. The SPNEs are

$$SPNE_1 = \{(Assert, Match), (Challenge, B \leq C_M)\}$$

$$SPNE_2 = \{(Assert, Don'tMatch), (Challenge, B > C_E)\}$$

- Suppose that Maria values the painting less than Elena does, i.e. $C_M < C_E$. At the last stage Maria will choose to match Elena's bid if $C_M > B$ and not to match Elena's bid if $C_M < B$. In the first case Elena's payoff is $-F$, while in the second case it will be $C_E - B$. The latter is positive if $C_E > B$. Elena will bid $B \in [\, C_M, C_E)$ and the painting will not end up with Maria. Therefore, there is no SPNE where Maria gets the painting if $C_M < C_E$.

(b) Consider now the case where Elena is the owner.

- Suppose that Elena values the painting more than Maria does, i.e. $C_E > C_M$. At the last stage Maria will choose to match Elena's bid if $C_M > B$ and not to match Elena's bid if $C_M < B$. In the first case Elena's payoff is $-F$, while in the second case it will be $C_E - B$, which is positive since $C_E > B$. Hence at her decision node Elena will choose to challenge and bid any amount B such that $C_E > B > C_M$. Anticipating this, at her first decision node Maria will give up (and get a payoff of 0), because if she asserted her claim then her final payoff would be negative. Therefore, the painting goes to Elena, the legal owner. The payoffs are 0 for Maria and C_E for Elena. The SPNE is

$$SPNE_4 = \{(Give\ up, Don'tMatch), (Challenge, C_M < B < C_E)\}$$

- Suppose that Elena values the painting less than Maria does, i.e. $C_E < C_M$. At the last stage Maria will choose to match Elena's bid if $C_M > B$ and not to match Elena's bid if $C_M < B$. In the first case Elena's payoff is $-F$, while in the second case it will be $C_E - B$, which is negative since $C_E < C_M < B$. Hence at her decision node Elena will choose not to challenge. Anticipating this, at her first decision node Maria will assert the claim. In this case it is not possible for Elena to get the painting (despite being the legal owner) should she value the painting less that Maria does.