Faustmann's Formulas for Forests

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Abstract. The canonical, Faustmannian forest is revisited to sharpen understanding of forests as forms of capital. Investment and two r-percent rules are discussed and re-interpreted. A forest's two natural resources, the stand and the land, act in conjunction as a composite asset. Non-marketed or intangible capital is also absorbed into the single composite. If capital is comprehensively defined, there is no independent role for the concept of an internal rate of return. A forest provides real options in optimal and sub-optimal rotation patterns. Old growth has superficial similarities to a exhaustible resource, but the forest still consists of two resources that, in conjunction, behave comparably to a plantation forest.

Key words: comprehensive investment, r-percent rule, internal rate of return, real options, intangible capital

JEL Classification: Q23

1. INTRODUCTION

Historically, a stylized plantation forest has been a main vehicle for developing the theory of capital. A remarkable analysis was provided in 1849 by Martin Faustmann. The contribution of forestry to economics, and vice versa, is a theme in Gane's (1968) examination of papers by Faustmann (1968) and Faustmann's contemporary, E.F. von Gehren (1968).

Although there now exist sophisticated economic models of forests, the familiar, stylized, canonical model brings out the specific features of a forest the most sharply for interpreting it as a form of capital. Each of two dissimilar equations is sometimes called Faustmann's formula by students of forest economics. What may be called Faustmann's *formulas* formalize the broad considerations involved in the harvest decision. They provide a link, through the prevailing rate of interest and other prices, of the forest to other assets and the rest of the economy.

One of the formulas applies to both optimal and non-optimal forest management and affords a simple comparison of the two. It is possible to consider the implications of an insightful forester's recognizing that maximum present value should be pursued rather than a sub-optimal rotation. The discussion stresses that a forest is a single asset made up of two natural resources, namely (1) the growing trees or *stand* and (2) the *land*, which comprises land area, soil, climate and general capacity to grow a stand. Both resources contribute in concert, and not individually, to the realization of value. They remain a composite even in the special situation of old growth.

Faustmann's two formulas, which are the foundation of the present paper, are well known. Re-interpretations of them are presented as two theorems about capital value. A number of further results that add to the re-interpretations are presented as corollaries to these theorems.

2. TWO RESOURCES, ONE ASSET

Consider a forest under stationary conditions and under certainty. Because conditions are stationary, among other things the interest rate, r, is taken to be constant through time. Also, the age of harvest must be the same in each rotation.

Faustmann (1968) only briefly mentions the *choice* of rotation age. His main preoccupation is with a forest for which, in modern terminology, a particular *resource-allocation mechanism* is in place (Dasgupta and Mäler 2000). In this case the mechanism predetermines a rotation age. That age is not necessarily optimal. Let it be represented by $a > 0$, the *net* revenues of the harvest by $R(a)$ and the planting cost by c.

A forest may also provide flows of thinnings, amenity values and costs, $\alpha_i(t), i = 1, ..., n$, at any age $t \in (0, a)$ (e.g. Faustmann 1968, Hartman 1976, Strang 1983, Pearce 1994). Let the set of indices of *internalized* flows, identified as amenities or simply flows, be denoted by $I \subseteq \{1, 2, ..., n\}$. A flow of cost, for maintenance or administration, is a negative value. Only the algebraic sum of the internalized flows is used below. Therefore, let $\alpha(t) = \sum_{i \in I} \alpha_i(t)$.[1] Also let $\int_0^t \alpha(s) e^{-rs} ds = A(t)$ denote the present value of internalized flows up to age $t \leq a$. Let $F(a)$ be defined to be the sum of revenues from harvesting at age a and the cumulated, internalized values of the flows:

$$F(a) = R(a) + e^{ra} A(a).$$

Most writers, including Faustmann, have concerned themselves with determining the value of the land. If the rotation age is predetermined to be a, the present value

[1] At the level of abstraction of this paper, a *stand* can be any collection of trees of the same age and species, on contiguous or dispersed parcels of land, under the management of a price-taking forester. It is assumed that any jointness in provision of the internalized flows is consistently attributed to the collections. A non-uniform forest can be viewed as akin to a portfolio of such stands.

of bare forest land is

$$\begin{aligned} L(a) &= -c + \frac{F(a)}{e^{ra}} - \frac{c}{e^{ra}} + \frac{F(a)}{e^{2ra}} - \frac{c}{e^{2ra}} + ... \\ &= -c + \frac{F(a)}{e^{ra}} + \frac{1}{e^{ra}} \left[-c + \frac{F(a)}{e^{ra}} - \frac{c}{e^{ra}} + \frac{F(a)}{e^{2ra}} - \frac{c}{e^{2ra}} + ... \right] \\ &= -c + \frac{F(a)}{e^{ra}} + \frac{1}{e^{ra}} L(a) . \end{aligned}$$

The following were derived by Faustmann (1968: 30):

$$L(a) = \frac{F(a) - ce^{ra}}{e^{ra} - 1}; \tag{1}$$

$$L(a) = -c + \frac{F(a) - c}{e^{ra} - 1}. \tag{2}$$

According to equations (1) and (2) the value of the forest is discounted into the land value, $L(a)$, no matter what the value of a. If $L(a) < 0$, or equivalently if $F(a) - ce^{ra} < 0$, for every a, the forest is not planted. In an interesting problem there is an a_0, depicted in Fig. 1, such that $F(a_0) - ce^{ra_0} = 0$, and $a > a_0$ for a forest that is planted. For the rotation age a_0 the value of the land, $L(a_0)$, is zero.

Equation (2) can be rearranged to reveal analytic features of the problem beyond solving for the value of the land. For any fixed rotation age $a > a_0$,

$$L(a) + c = \frac{F(a) - c}{e^{ra} - 1}$$

and

$$\begin{aligned} F(a) + L(a) &= F(a) - c + \frac{F(a) - c}{e^{ra} - 1} \\ &= e^{ra} \frac{F(a) - c}{e^{ra} - 1} \\ &= e^{ra} (L(a) + c) . \end{aligned} \tag{3}$$

During a rotation, the value of the land is not recovered. At the end of the rotation, the land is returned intact, along with the harvested timber. Formula (3) expresses the following *r-percent growth rule*.

Theorem 1 Faustmann's General Formula. *Let the resource-allocation mechanism set a predetermined, repeated rotation age that is not necessarily optimal. The sum of the value of bare land and the planting cost grows at the rate of interest to the sum of the values of (a) the harvest, (b) the cumulated, internalized amenities and costs and (c) the bare land.*

Formula (3) is considered *general* because it applies to both an optimally and a non-optimally exploited forest. Although it is the formula explicitly examined by Faustmann (1968), it has not been stressed in the economics of forestry. But it does help in the interpretation of the economic properties of forest assets.

An implication of formula (3) is that *investment* in the forest includes the value of the contribution of the bare land, in kind, in addition to the planting cost. Even though it has no alternative use, bare land has an opportunity cost: there is a choice of the planting time. The bare land is an asset committed at age $t = 0$ to the enterprise over the rotation period.

Corollary 2 Investment. *The investment in a forest is the sum of the planting cost and the value of the land.*

To stress the comprehensiveness of forest capital, let the total value invested be written

$$\Phi(a) = c + L(a).$$

According to formula (3), the total investment $\Phi(a)$ grows at rate r to the total value obtained at the end of the rotation (including the cumulated value of flows of amenities and costs):

$$\Phi(a) e^{ra} = [R(a) + A(a) e^{ra}] + L(a) = F(a) + L(a).$$

The r-percent growth rule (3) is not explicit in Faustmann's analysis or in other analyses such as Samuelson's (1976) or Hartman's (1976). Given the chosen rotation period the total investment earns a rate of return of r.

Corollary 3 Internal rate of return. *(i) The* internal *rate of return on the total value invested in a forest, including the value of the land, is equal to the rate of interest. (ii) A finding that the internal rate of return is greater than the rate of interest is indication that forest capital has not been defined comprehensively.*

Conventionally, the internal rate is given by the solution, ρ, to the equation $c(1+\rho)^a = F(a)$. The internal rate is used as a criterion of investment by not considering the forest land to be invested, as if it is not scarce (has no value). Once investments and yields are *comprehensively* defined to include the land, the internal rate of return is redundant: total investment $\Phi(a)$ grows at the external (market) rate r to $F(a) + L(a)$.

Faustmann and Samuelson define the (full) land rental to be $rL(a)$. For any rental charge no greater than this value the rotation a can be supported, with a non-negative profit accruing to the forester. Even if there is a rental contract for the use of the land, the value $c + L(a)$ is invested through the forester's choice and is paid back with interest. Computation of the full rental, $rL(a)$, presupposes computation of $L(a)$. The essence of the analysis is the role of the land and stand as capital stocks, not how the forest is financed. Fig. 1 depicts the value of a forest with chosen rotation (resource-allocation mechanism) a_1.

Faustmann (1968) insists that the land value remains $L(a)$ at all ages. The land would have value $L(a)$ at age $t \in (0, a)$ if it were bare. But it is not bare. During a rotation the land is not available for sale independently. Faustmann's contemporary, Gehren (1968: 24), comes closer to realizing the point: "At the time of planting, the acre becomes forest." On the other hand, Gehren wishes to assign a portion of the total value to the land.

The interplay of land and stand is fundamental to formula (3). Once invested, the land and stand form a composite, a *forest*. The value of the land can be realized only by cutting the stand. The value of the stand can be realized only by restoring the bare land.

Corollary 4 *A forest with standing trees is a single asset that comprises two natural resources, the stand and the land.*

The change in perspective on forest capital is definitional, not tautological: there is a wider, comprehensive definition of capital. The land is transformed to a forest by planting. The land and the planting cost remain invested until the harvest age a. At age a the forest is transformed to the harvest with value $R(a)$ and bare land with value $L(a)$. Amenities accrue at rate $\alpha(t)$. The cumulated value of these flows is a part of what is returned, in addition to the value of the harvest. The flows are also attributable to the forest and not to either natural resource alone.

3. OPTIMAL MANAGEMENT

The optimal rotation age $\hat{a}$, which is assumed to be unique, maximizes the net present value (discounted cash flow) of the forest:

$$L(\hat{a}) = \max_a \frac{F(a) - ce^{ra}}{e^{ra} - 1} > 0. \tag{4}$$

Setting $L'(\hat{a}) = 0$ yields that

$$F'(\hat{a}) = \frac{F(\hat{a}) - c}{e^{r\hat{a}} - 1} re^{r\hat{a}}; \tag{5}$$

$$F'(\hat{a}) = r(F(\hat{a}) + L(\hat{a})). \tag{6}$$

Sometimes equation (6) is considered to be Faustmann's formula. Faustmann (1968) makes allusions to the possibility of doing better than following the resource-allocation mechanism that is in place. Although it has been emphasized in the economics of forestry, Faustmann does not solve for the optimal rotation.

Another way to express equation (6) is to let $V(a) = F(a) + L(a)$. Since $L'(\hat{a}) = 0$,

$$V'(\hat{a}) = F'(\hat{a}) + L'(\hat{a}) = F'(\hat{a}) = r(F(\hat{a}) + L(\hat{a})) = rV(\hat{a}). \tag{7}$$

Formula (7) is the expression of another r-percent growth rule (cf. Cairns and Davis 2007: 470).

Theorem 5 Faustmann's Special Formula. *At the* optimal *strike point, the total, cumulated, internalized value of the forest (land value plus current internalized value) is growing at the rate of interest.*

This exact r-percent rule is depicted in Fig. 2 as the tangency at $a = \hat{a}$. Formula (7) is specific to an optimally managed forest. It, too, can be re-interpreted.

If forest land is not scarce, prices adjust through competition such that $L(\hat{a}) = 0$. Wicksell's formula for optimal rotation-by-rotation exploitation is an immediate special case.

Corollary 6 Wicksell's formula. *If* $L(\hat{a}) = 0$ *then*

$$F'(\hat{a}) = rF(\hat{a}).$$

Equation (5) is equivalent to Hartman's equation (9) or (10). However, Hartman (1976: 57) expresses his r-percent rule as a rule for the increase in stumpage value and formulates it using a "correction factor" to the rate of interest. The interest rate in Faustmann's model is a parameter taken by the forester. It is a required return, earned on the entire investment. A rate of interest, even corrected, applied solely to the stand neglects the investment of the land and is appropriate only when Corollary 6 holds (only when land is not scarce).

Corollary 7 *An "adjusted" rate of return that exceeds the market rate is an indication that forest capital is not being defined comprehensively.*

Formula (7) states that, at the optimal strike point, value, appropriately defined, rises at exactly the rate of interest. Consider the expression $F'(t) / [F(t) + L(\hat{a})]$.

Since $L'(\hat{a}) = 0$, the expression is the rate of change of the cumulated internalized value (from immediate cutting) of the forest. It is equal to r at $\hat{a}$, and is a decreasing function at $\hat{a}$. (See the appendix.) A general result found by Davis and Cairns (2011) applies to the forest.

Corollary 8 *The rate of increase of cumulated internalized value is greater than the rate of interest before and less than the rate of interest after the optimal harvest age.*

The optimal rental for this forest is $rL(\hat{a})$; it leaves no profit in the hands of the forester. If a fixed annual rental is charged that is less than $L(\hat{a})$, there is no change to the optimal rotation period.

A change in a stumpage charge (for harvested timber) affects the rotation by affecting the value of $F(a)$. Both the rate of growth and level of $F(a) + L(a)$ are affected by a shift of $F(a)$. As has been found by many authors,

$$\frac{1}{F(a) + L(a)} \frac{dF(a)}{da} |_{a=\hat{a}} < \frac{1}{F(a)} \frac{dF(a)}{da} |_{a=\hat{a}} .$$

An explicit consideration of the land as a resource shortens the rotation as compared to considering only the stand as a resource, with the harvest age being optimized rotation by rotation as in Wicksell's formula.

Example 1. One amenity may be to fix carbon at rate $\alpha_C(t)$, $t < a$ (cf. Pearse 1994). At ages $t > a$ the harvested wood may be burnt or decay and return carbon to the atmosphere at rate $\beta_C(t, a)$. (See Kooten, Binkley and Delcourt 1995; Ariste and Lasserre 2001; Cairns and Lasserre 2004, 2006.) Consistently with the assumption of stationarity, let carbon have a constant price p in the permit market. Then the present value of the net contribution of carbon of the forest for rotation period a is

$$A_C(a) = \int_0^a p\alpha_C(t) e^{-rt} dt - \int_a^\infty p\beta_C(t, a) e^{-rt} dt.$$

Since $\int_0^a \alpha_C(t) dt = \int_a^\infty \beta_C(t, a) dt$, it is easily seen that, because of "the power of compound interest", $A_C(a) > 0$: The contribution of a single rotation of a stationary forest to mitigating climate change is positive, even though all of the carbon is eventually returned to the atmosphere.

If there are only two sources of value, the harvest value and the value of fixing carbon, then

$$\frac{\frac{d}{da}\left(F\left(\hat{a}\right)+L\left(\hat{a}\right)\right)}{F\left(\hat{a}\right)+L\left(\hat{a}\right)}=\frac{\frac{d}{da}\left(R\left(\hat{a}\right)+L\left(\hat{a}\right)\right)+\frac{d}{da}\left[e^{ra}A_C\left(\hat{a}\right)\right]}{\left(R\left(\hat{a}\right)+L\left(\hat{a}\right)\right)+e^{ra}A_C\left(\hat{a}\right)}.$$

At the harvest age $a_{\setminus C}$ that would be optimal if the carbon credits were neglected,

$$\frac{\frac{d}{da}\left[e^{ra}A_C\left(a_{\setminus C}\right)\right]}{e^{ra}A_C\left(a_{\setminus C}\right)}=r+\frac{p\left[\alpha_C\left(a_{\setminus C}\right)+\beta_C\left(a_{\setminus C},a_{\setminus C}\right)\right]}{A_C\left(a_{\setminus C}\right)}>r=\frac{\frac{d}{da}\left(R\left(a_{\setminus C}\right)+L\left(a_{\setminus C}\right)\right)}{R\left(a_{\setminus C}\right)+L\left(a_{\setminus C}\right)}.$$

Therefore,

$$\frac{\frac{d}{da}\left(F\left(a_{\setminus C}\right)+L\left(a_{\setminus C}\right)\right)}{F\left(a_{\setminus C}\right)+L\left(a_{\setminus C}\right)}>r:$$

under stationary conditions, an internalization of the value of carbon fixing increases the optimal harvest age.

4. OPTION

Faustmann (1968: 27) chastises Gehren for evaluating an “immature” stand (one of age less than the harvest age under the prevailing resource-allocation mechanism) at its immediately realizable value rather than discounting the value that can be anticipated if the forest grows to maturity. In the present notation, the distinction is between $R\left(t\right)+L\left(a\right), t<a$ and the value,

$$\left[F\left(a\right)+L\left(a\right)\right]e^{-r(a-t)}-A\left(t\right)e^{rt}=\left[c+L\left(a\right)-A\left(t\right)\right]e^{rt},$$

that remains in the forest (that has not been received by age t through the flows).

Gehren (1968) did not even consider the possibility of deviating from his specified rotation age of eighty years, and as a result came to a contradiction that he failed to resolve. Faustmann (1968) pointed out that eighty years was too long, that 65 years was a more profitable choice. While he mentioned an optimal rotation, he did not take the step of finding it.

The value that remains is the value of the immature forest: As is noted by Samuelson (1976), if the forest is sold as a whole the seller can realize $[c + L(a) - A(t)] e^{rt}$. If the two resources are sold piecemeal, $R(t) + L(a)$ can be realized. Is the latter greater than the former? Is the whole greater than the sum of the parts?

Consider rotations that are *shorter than optimal.* In this case, Faustmann is right that, under the prevailing resource-allocation mechanism, the forest should be valued at the discounted, remaining value. (See Fig. 1.) The difference, $[F(a) + L(a)] e^{-r(a-t)} - A(t) e^{rt} - [R(t) + L(a)] > 0$, can be realized only if the forest is allowed to grow to "maturity" at a.

Even though the analysis is under certainty, this difference depends on the timing of harvest. It is the value of an *option* to harvest at a rather than at t. "Accept the most profitable irreversible investment if and only if its current return exceeds the value of the options thus forfeited" (Bernanke 1983: 90). The option for this asset undergoing gestation is a part of the value of the single asset.

Corollary 9 Option. *There is an option to cut at any age during the rotation. The option value of the forest at an age before harvest is the difference between two values. One is the value realizable at the chosen harvest age, discounted to t and net of the flows already obtained. The other is the so-called intrinsic value, the sum of the value of bare land and the net revenue that would be earned from harvesting at t. If the forest is sold as a whole, the value realized includes the option value.*

Under the resource-allocation mechanism (harvest at age a), the value of the forest at $t < a$ is not the net realizable value, but the value (i) inclusive of the option and (ii) net of the cumulative value of any amenities received so far, namely, $\Phi(a) e^{rt} - A(t) e^{rt}$. This value would be received in a market for the forest with trees of age t. The value of the option at age a is

$$\begin{aligned}\Omega(t,a) &= \Phi(a)\,e^{rt} - A(t)e^{rt} - [R(t) + L(a)] \\ &= \Phi(a)\,e^{rt} - A(t)e^{rt} - \left[F(t) - A(t)e^{rt} + L(a)\right] \\ &= \Phi(a)\,e^{rt} - [F(t) + L(a)]\,. \end{aligned} \tag{8}$$

The analysis of harvesting as an option indicates that there is no need for a separate analysis for an optimally managed forest that has age $t \in (0, \hat{a})$. (See also Strang 1983.) The optimal harvest age remains $\hat{a}$ and requires waiting for $\hat{a} - t$ periods before cutting (until the value of the option reaches zero). The decision to wait is made at each age $t \in (0, \hat{a})$.

Even though the decision to harvest is with respect to the age of the trees only, since the land remains invested while the stand is growing the option value applies to the composite of stand and land, to the forest. Option value does not apply to the realizable value of the trees alone. The sum of the realizable values of the trees and the land is less than the market value:

$$R(t) + L(a) < R(t) + L(a) + \Omega(t,a)\,.$$

Since the value remaining in the forest (net of flows already realized) is $[\Phi(a) - A(t)]\,e^{rt}$ at age $t < a$ and harvesting at t yields $R(t) + L(a)$, the following helps to give some precision to the sometimes vague notion of what capital value is sunk in the forest during the gestation period.

Corollary 10 Sunk Value. *The value that is sunk in the forest at any age during a rotation is the option value.*

Option value arises under certainty and also under a non-optimal resource-allocation mechanism with $a < \hat{a}$. Discussion of option value is more familiar under optimality. For $t < \hat{a}$, the option has a familiar look, but in the time domain.

5. EXAMPLE 2: OLD GROWTH

Let old growth be defined as a state in which the forest is no longer growing. This definition broadens the notion of forest primeval to include a forest that has grown in a plantation. The analysis herein applies to both. Suppose that the state of old growth begins at age $\bar{a} > \hat{a}$. Suppose also that the forest is not subject to fire or pestilence. Then, for $t \geq \bar{a}$, $R(t)$ and $\alpha(t)$ are constant at $\bar{R} = R(\bar{a})$ and $\bar{\alpha} = \alpha(\bar{a})$.

There is an option to harvest old growth or not to to harvest it. A necessary and sufficient condition for *not* harvesting is that

$$\int_0^\infty \bar{\alpha} e^{-rt} dt = \frac{\bar{\alpha}}{r} > \bar{R} + L(\hat{a}). \tag{9}$$

(For a primeval forest, total amenities are almost surely higher than for a harvested forest that has achieved its maximum size, so that future harvesting is less likely.) In addition, Strang (1983) shows that there is an age $a^* < \bar{a}$ such that, if the current age $t \in (\hat{a}, a^*)$, the forest is harvested immediately and if $t > a^*$ the forest is never harvested.

In an old-growth forest there are still two natural resources, the stand and the land, but a single asset. There are again joint products if the forest is harvested (if the above inequality is reversed), like wool and sheared sheep; the total value is the sum of the realizable values of the trees and the land. Because the forest is past the optimal harvest age, the option value is zero.

As with an exhaustible resource, when demand is perfectly elastic and stationary and there is no constraint on the rate of cutting, extraction of old growth is immediate. Unlike an exhaustible resource, however, an old-growth forest has a positive opportunity cost of cutting, $\bar{\alpha}/r$. This feature of old growth provides a disincentive to cut that does not exist for a canonical exhaustible resource. As with other amenity flows, it is attributable to the forest and not to either of land or stand.

Moreover, an old-growth stand is exhaustible but forest land is not. In a canonical exhaustible-resource model the land has an opportunity cost of zero. Old growth is an exhaustible resource only if the land has no use once the old-growth stand is harvested.[2] An analogy is pertinent here. If an immature forest is to be cut so that the land can be used for some other use, are the immature trees exhaustible? Let the value of the land in the better use be $L^* > L(\hat{a})$. If

$$\frac{1}{R(a)+L^*}\frac{d(R(a)+L^*)}{da} = \frac{1}{R(a)+L^*}\frac{dR(a)}{da} < r,$$

the indication is to cut immediately. Otherwise one waits until $[1/(R(a)+L^*)]\,dR(a)/da = r$, a condition analogous to equation (7).

Old growth may be harvested *over time* if its demand slopes downward but demand for new growth is elastic.

- Let $q_s \in 0,1)$ represent the fraction of the old growth that is harvested at time s and $Q_t = \int_0^t q_s ds \in [0,1]$ represent the cumulative harvest to time t.
- Let the harvest be on the interval $[0,T]$, such that $Q_T = 1$.
- Let net cash flow be represented by $\pi(q_s)$.

The land on which the harvested trees stood is replanted as the forest is harvested and has value $q_s L(\hat{a})$. The problem of the forester responsible for an old-growth forest at time $t \in [0,T)$ is to maximize

$$V(t) = \int_t^T \pi(q_s)\,e^{-r(s-t)}ds + \int_t^T q_s L(\hat{a})\,e^{-r(s-t)}ds + \int_t^T \bar{\alpha}(1-Q_s)\,e^{-r(s-t)}ds.$$

The current-value Hamiltonian for the problem is

$$H(q,\mu,Q) = \pi(q) + qL(\hat{a}) + \bar{\alpha}(1-Q) + \mu q.$$

[2]If old-growth trees can be replaced by letting newly planted trees grow to $\bar{a}$ then the forest is as renewable as any plantation forest. If not, then one must see the forest as being more than trees.

The Hamiltonian is the sum of the contributions of the two resources, land and stand, which still form a single asset in keeping with Faustmann's formulas. The optimality conditions, $\partial H/\partial q = 0$ and $\dot{\mu} = r\mu - \partial H/\partial Q$, imply that, on an optimal path,

$$\int_0^t \bar{\alpha} e^{r(t-s)} ds = -\mu_t = \pi'(q_t) + L(\hat{a}), t \in [0, T].$$

The shadow value μ_t is composed of contributions from the harvest and the land released by harvesting. Moreover, the amenities from old growth affect the evolution of the shadow price,

$$\frac{\dot{\mu}_t}{\mu_t} = \frac{1}{\pi'(q_t) + L(\hat{a})} \frac{d[\pi'(q_t) + L(\hat{a})]}{dt} = r + \frac{\bar{\alpha} Q_t}{\pi'(q_t) + L(\hat{a})}. \tag{10}$$

The first equality, for an infinitesimal unit of the forest, is comparable to Faustmann's Special Formula for an optimal forest at the point of harvest, formula (7).

Because of the additional opportunity costs of an old-growth forest, formula (10) differs from Hotelling's rule for a homogeneous, exhaustible resource, which for shadow value ν_t would be

$$\frac{\dot{\nu}_t}{\nu_t} = \frac{\dot{\pi}'(q_t)}{\pi'(q_t)} = r.$$

For an exhaustible resource, $\bar{\alpha} = 0$ and $L(\hat{a}) = 0$.

Corollary 11 Faustmann's Special Formula, Old Growth. *Like any forest, an old-growth forest is a single asset that encompasses two resources. Although old growth is exhaustible, an old-growth forest is not an exhaustible resource, because of two additional opportunity costs: Each unit of old growth harvested releases a unit of land and each unit remaining continues to provide amenities. The r-percent rule is a special case of Faustmann's Special Formula for optimal management.*

The r-percent rule holds that growth in the shadow value of old growth must make up for a negative contribution, the loss of amenity value:

$$\frac{\dot{\mu}_t}{\mu_t} - \frac{\bar{\alpha} Q_t}{\pi'(q_t) + L(\hat{a})} = r.$$

An obvious modification holds if the use of the land after harvest is modified, such as for agriculture or housing development.

6. MANAGEMENT AS AN INTANGIBLE ASSET

Optimal management consists of maximizing the value of the forest, which is equivalent to maximizing the value of the forest land (finding $\hat{a}$ to maximize $L(a)$) at the beginning of a rotation. If a different harvest age, $\tilde{a}$ say, is chosen, value is lost.

Equations (1), (2) and (3) hold for any value of a, not just for $\hat{a}$ (not just for an optimum). That is to say, if the resource-allocation mechanism in place prescribes that $a = \tilde{a} \neq \hat{a}$ (such as Gehren's eighty years), then the investment at age 0, $\Phi(\tilde{a})$, grows at rate r to $L(\tilde{a}) + F(\tilde{a})$ (total value, given the choice of the rotation period) at age $\tilde{a}$. The value (and the market price) of bare forest land in the context of the resource-allocation mechanism is $L(\tilde{a})$.

Suppose that $\tilde{a} > \hat{a}$ and that the resource-allocation mechanism, followed by all foresters in the industry, is to harvest at age $\tilde{a}$. In this case, Faustmann's admonishment to Gehren, that the forest should be valued at its discounted remaining value from $\tilde{a}$, is not correct. Faustmann is right if $\tilde{a} < \hat{a}$.

The resource-allocation mechanism can be interpreted as being one that *seems* to most foresters as being incontrovertible. The ability to recognize an option that others do not recognize is an asset. (Think of the legendary figures of securities lore.) If that ability is scarce it commands a positive rent. If it is not scarce, as is implicitly assumed in the analysis of optimal management, its rent is zero. The option is to choose the most advantageous age to harvest from among a set of alternatives, and the decision may be to stop immediately rather than to continue to follow a convention that does not maximize value. In particular, the optimal policy is to wait until $t = \hat{a}$ if $t < \hat{a}$ or to cut immediately if $t > \hat{a}$ (subject to Strang's qualification). If $t > \hat{a}$, the option value if one forester among many recognizes the option for a potential increase in value is

$$\omega(t, \hat{a}) = [L(\hat{a}) + F(t)] - [L(\tilde{a}) + F(\tilde{a})] e^{-r(\tilde{a}-t)} > 0.$$

Corollary 12 *For resource-allocation mechanisms that specify rotations that are not optimal, the optimal policy is (i) if the age of the stand is less than the optimal rotation length to wait or (ii) if is greater, to cut immediately. Option value is the difference between the values of the optimal and the specified policies.*

In the context of conventional management, the gain can be seen to be the benefit of optimal management or of entrepreneurship. At any age $t < \tilde{a}$, the *shadow value* of good management is this option value. This shadow value is labeled "σ" in Fig. 2. In a perfect market it is paid to the forester with vision.

Vision or entrepreneurship is an *intangible* or non-marketed asset. Traditionally such assets have been neglected in capital theory. However, non-marketed capital is of increasing importance (Abraham 2005). Vision is a residual, the difference between optimal and common management. Faustmann's General Formula (Theorem 1) applies to both a well managed and a poorly managed forest. A well managed forest is a single, composite asset comprised of land, stand and management. The contribution of management is not independently observable; it can be derived only by a counterfactual analysis.

It follows from Faustmann's interchange with Gehren that a forester may have vision without being able to determine the optimal rotation. The improvement from Gehren's rotation of eighty years to Faustmann's of 65 is also an option (not necessarily the optimal option). The value of that option is also an intangible asset, a part of the composite value of the forest expressed as a residual.

Corollary 13 Intangible Assets. *Intangible assets, such as entrepreneurship, and tangible assets combine to form a single productive asset, the forest. The intangible asset improves results. It is measured as a residual and cannot be measured directly.*

Total investments in each of the superior and inferior programs earn the prevailing rate of interest.

7. CONCLUSION

Although Faustmann's analysis is well over a century and a half old, it is deep enough to enable a re-interpretation of the exploitation of a forest in terms of modern economic concepts.

For either optimal or non-optimal management of a stylized forest, there are two natural resources, stand and land. The combination of those resources is a composite asset that returns the going rate of interest over the chosen exploitation period. Other assets may also be involved; the paper has considered a scarce, intangible, non-marketed asset, management or entrepreneurship. Since each asset's internal rate of return is exactly the interest rate, there is, in the forest at least, no need for this concept when all assets are enumerated comprehensively.

Forestry is at its base a repeated point-input, point-output problem. The fundamental decision is the timing of the harvest and the implications of timing for capital value. In an optimal or non-optimal program, there is a real option throughout the rotation period. Even though the analysis is under certainty, there is a choice to wait or to strike. In particular, there may be an option to cut early and to obtain the sum of the realizable values of the two assets. The option value applies to the forest and is not attributable to any one of the resources that make up the forest. It provides the incentive to the decision maker to make an optimal intertemporal decision.

APPENDIX. RATE OF CHANGE OF TOTAL INTERNALIZED VALUE.

Equation (1) states that

$$L(a) = \frac{F(a) - ce^{ra}}{e^{ra} - 1}.$$

Simple differentiation w.r.t. a yields that

$$\begin{aligned}
\left(e^{ra} - 1\right)^2 L'(a) &= \left(F' - rce^{ra}\right)\left(e^{ra} - 1\right) - re^{ra}\left(F - ce^{ra}\right) \\
&= F'\left(e^{ra} - 1\right) - re^{ra}\left(F - c\right) \\
&= 0 \text{ at } \hat{a}.
\end{aligned}$$

The derivative of the LHS is

$$\begin{aligned}
2\left(e^{ra} - 1\right) re^{ra} L' + \left(e^{ra} - 1\right)^2 L'' &= \left(e^{ra} - 1\right)^2 L'' \text{ at } \hat{a} \\
&< 0 \text{ for a unique solution.}
\end{aligned}$$

The derivative of the RHS is

$$\begin{aligned}
F''\left(e^{ra} - 1\right) + rF'e^{ra} - r^2 e^{ra}\left(F - c\right) - rF'e^{ra} &= F''\left(e^{ra} - 1\right) - r^2 e^{ra}\left(F - c\right) \\
&= \left(e^{ra} - 1\right)^2 L'' \text{ at } \hat{a} \\
&< 0 \text{ from above.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{da}\left(\frac{F'}{F + \hat{L}}\right) &= \frac{F''}{F + \hat{L}} - \left(\frac{F'}{F + \hat{L}}\right)^2 \\
&= \frac{F''}{F + \hat{L}} - r^2 \text{ at } \hat{a} \text{ by eq. (6)} \\
&< r^2 \frac{e^{ra}}{e^{ra} - 1} \frac{F - c}{F + \hat{L}} - r^2 \text{ from above} \\
&= \frac{r^2 e^{ra}}{\left(e^{ra} - 1\right)\left(F + \hat{L}\right)}\left[F + \hat{L} - e^{ra}\left(\hat{L} + c\right)\right] \\
&= 0 \text{ at } \hat{a} \text{ by eq. (3).}
\end{aligned}$$

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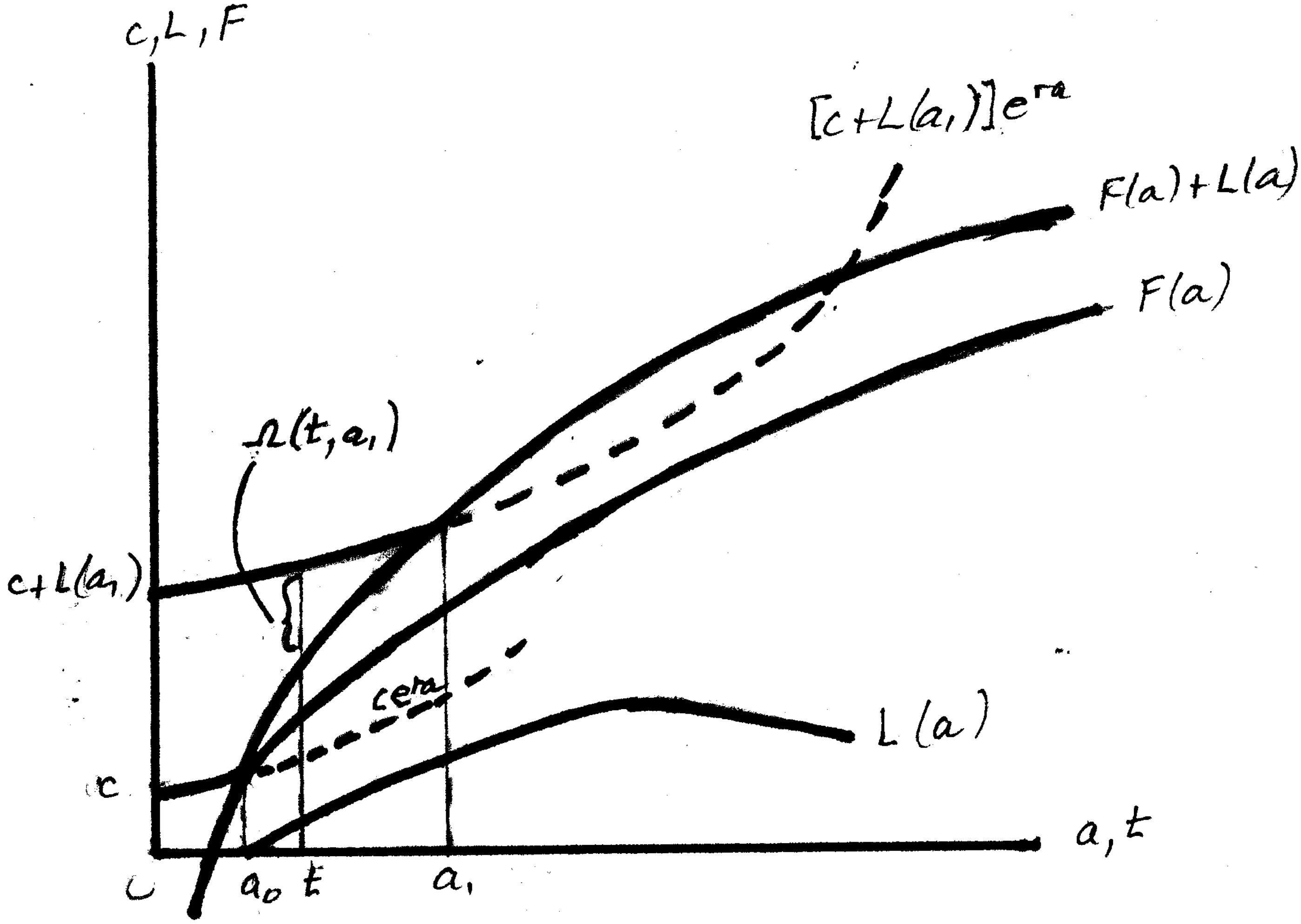

Figure 1. Faustmann's Formula No. 1

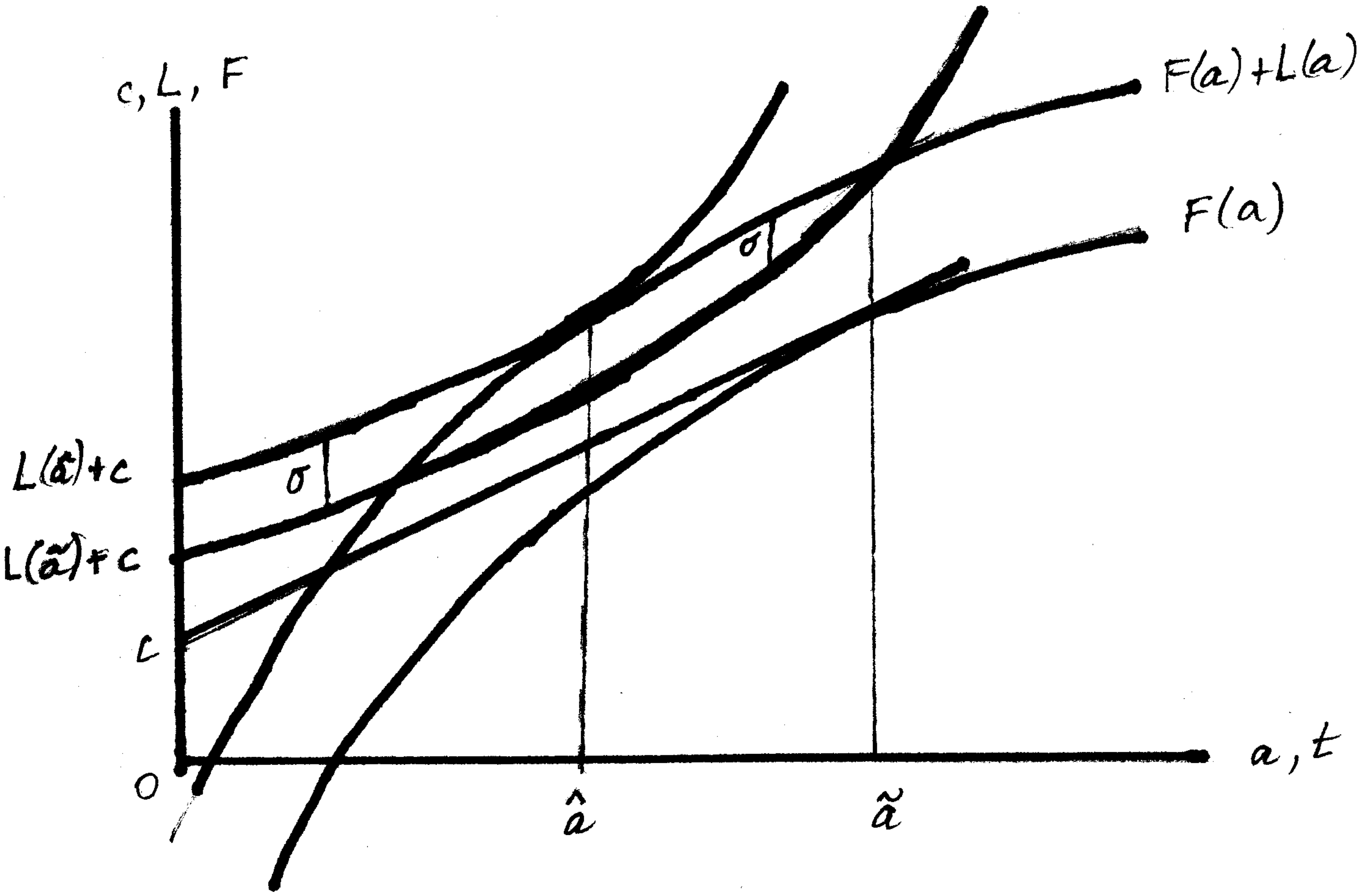

Figure 2. Faustmann's Formula No. 2. Maximum Sustainable Yield at ã vs Maximum Present Value at â.