GAME THEORY

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LECTURE NOTES SET 8: SEQUENTIAL GAMES OF INCOMPLETE INFORMATION

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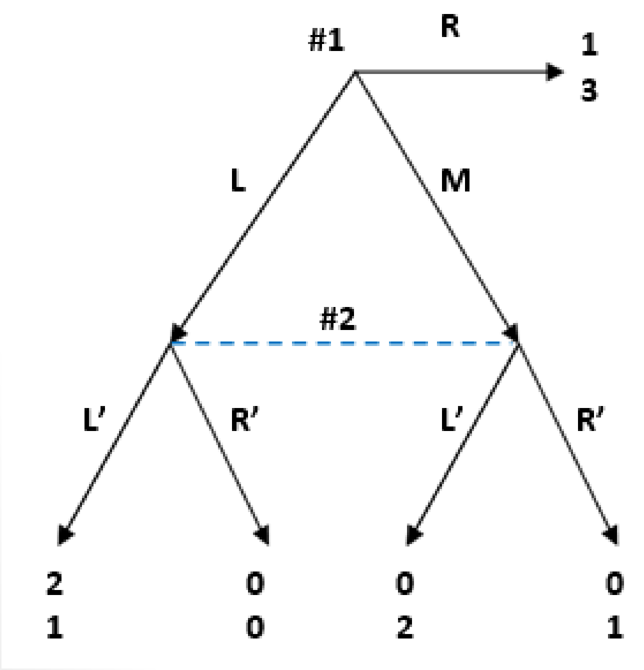
Extensive form games with incomplete information

In extensive games with imperfect information players may, at some point, face a situation inconsistent with the presumed equilibrium being played (i.e., find themselves on an off-equilibrium path). How should they behave there? Is sub-game perfection enough to correctly describe how players should behave off-equilibrium?

Example



#2

#1		L'	R'
	L	2,1	0,0
	M	0,2	0,1
	R	1,3	1,3

In this case NE=SPNE because there is only one sub-game (i.e., the entire tree). Note then that

$$SPNE_1 = NE_1 = \{L, L'\}$$

$$SPNE_2 = NE_2 = \{R, R'\}$$

But there is an inconsistency → player #2 will NEVER play R' as it is a strictly dominated strategy!

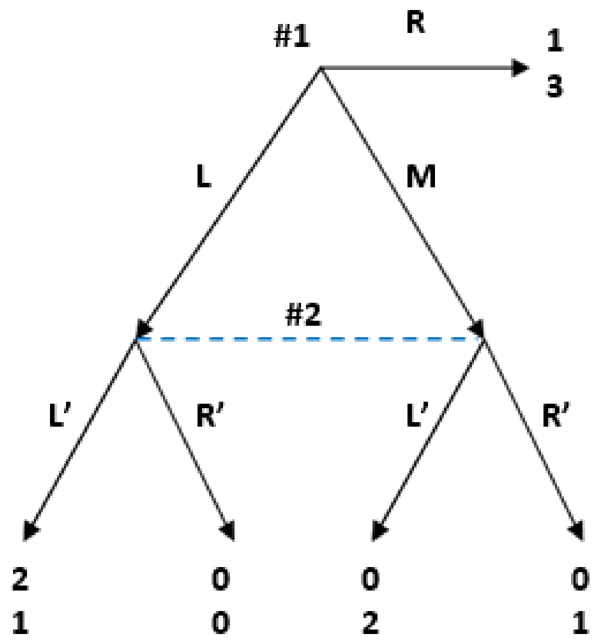
Extensive form games with incomplete information

We have to impose some requirements →

Requirement 1: at each information set the players with the move must have a belief (i.e., a probability distribution) over the nodes in the information set.

Requirement 2: players' strategies must be sequentially rational. At each information set the action taken by the players must be optimal given the players' beliefs and everyone's subsequent strategies.

In the previous example



#2, if he finds himself at his information set, he MUST assign probabilities over the nodes on that information set.

Then, #2 compares his expected payoffs from choosing one or the other action:

$$EU_2(L')=p\times1+(1-p)\times2=\ldots=2-p$$
$$EU_2(R')=p\times0+(1-p)\times1=\ldots=1-p$$

Extensive form games with incomplete information

Requirement 3: At information sets on the equilibrium path, beliefs are determined by Bayes' Rule.

Requirement 4: At information sets off the equilibrium path beliefs are determined by Bayes' Rule where possible.

Hence…

… A Perfect Bayesian Nash Equilibrium is a set of strategies and beliefs such that strategies are sequentially rational given the players' beliefs and players update their beliefs based on Bayes' Rule wherever possible.

Signaling games

- Extensive games of imperfect information where informed players move first
- A signaling game has (at least) two players
  ➔ A sender $\boldsymbol{S}$ of the signal
  ➔ A receiver $\boldsymbol{R}$
- Nature $\boldsymbol{N}$ draws type $\boldsymbol{t_i}$ for the sender from $\boldsymbol{T_S = \{t_1, t_2, \dots, t_n\}}$ according to a probability distribution $\boldsymbol{p(t_i) > 0}$ where $\boldsymbol{p(t_1) + \cdots p(t_n) = 1}$ (*i.e.*, the prior beliefs)
- The sender learns $\boldsymbol{t_i}$ and chooses a message $\boldsymbol{m_j}$ (action of $\boldsymbol{S}$) from $\boldsymbol{M = \{m_1, \dots, m_x\}}$
- The receiver observes $\boldsymbol{m_j}$ and chooses an action $\boldsymbol{a_k}$ from $\boldsymbol{A = \{a_1, \dots, a_y\}}$
- Payoffs are calculated by $\boldsymbol{u_S(t_i, m_j, a^*(m_j))}$ and $\boldsymbol{u_R(t_i, m_j, a_k)}$

Perfect Bayesian Equilibrium in Signaling Games

Requirement 1: after receiving any message $\boldsymbol{m_j} \in \boldsymbol{M}$, the receiver must have a belief about which types could have sent $\boldsymbol{m_j}$:

$$\mu(t_i|m_j) \geq 0, \quad \forall t_i \qquad \text{s.t.} \qquad \sum_{t_i \in T_S} \mu(t_i|m_j) = 1$$

Requirement 2:

Receiver → for each $\boldsymbol{m_j} \in \boldsymbol{M}$, the receiver's action $\boldsymbol{a^*(m_j)}$ must maximize the receiver's expected payoff given the belief $\boldsymbol{\mu(t_i|m_j)}$

$$a^*(m_j) = arg \max_{a^* \in A} \sum_{t_i \in T_S} \mu(t_i|m_j)\, u_R(t_i, m_j, a_k)$$

Sender → for each type $\boldsymbol{t_i} \in \boldsymbol{T_S}$ the sender's message $\boldsymbol{m^*(t_i)}$ must maximize the sender's payoff given the receiver's strategy (i.e., backward induction is implied here)

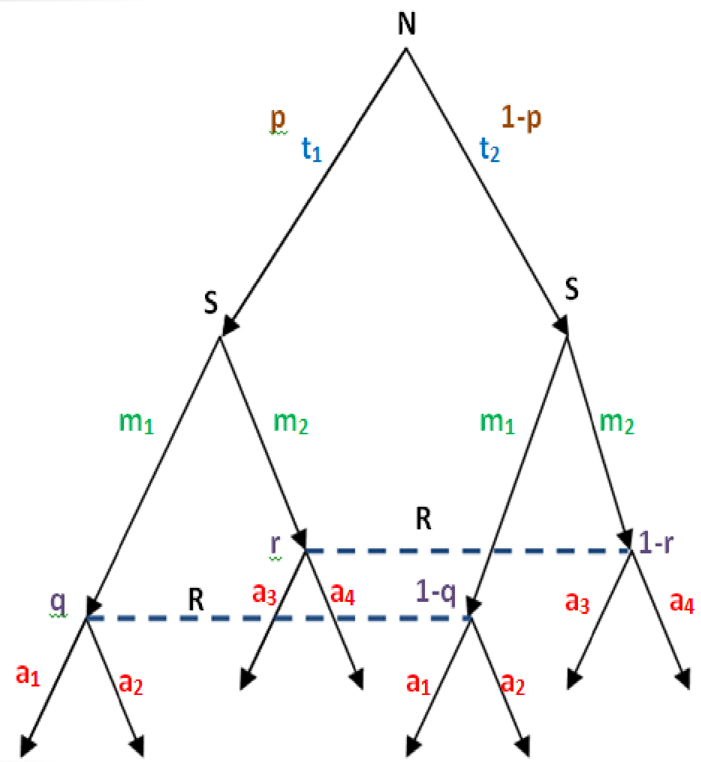
$$m^*(t_i) = arg \max_{m_j \in M} u_S(t_i, m_j, a^*(m_j)), \qquad \forall t_i$$

Requirement 3: for each $\boldsymbol{m_j} \in \boldsymbol{M}$ that is on the equilibrium path the receiver's beliefs must follow from Bayes' Rule and the Sender's strategy

$$\mu(t_i|m_j) = \frac{p(t_i, m_j)}{p(m_j)}$$

Perfect Bayesian Equilibrium in Signaling Games

Example



- Types: $T_S = \{t_1, t_2\}$
- Priors: t_1 with probability p and t_2 with probability $(1-p)$
- The Receiver assigns probabilities on each note on any of his given information sets, i.e., q and (1-q) on the nodes on the first information set, and r and (1-r) on the nodes on the second information set.
- Beliefs of R: the Receiver updates his beliefs (i.e., the probabilities on each node on any of his information sets) using Bayes' Rule. For example,

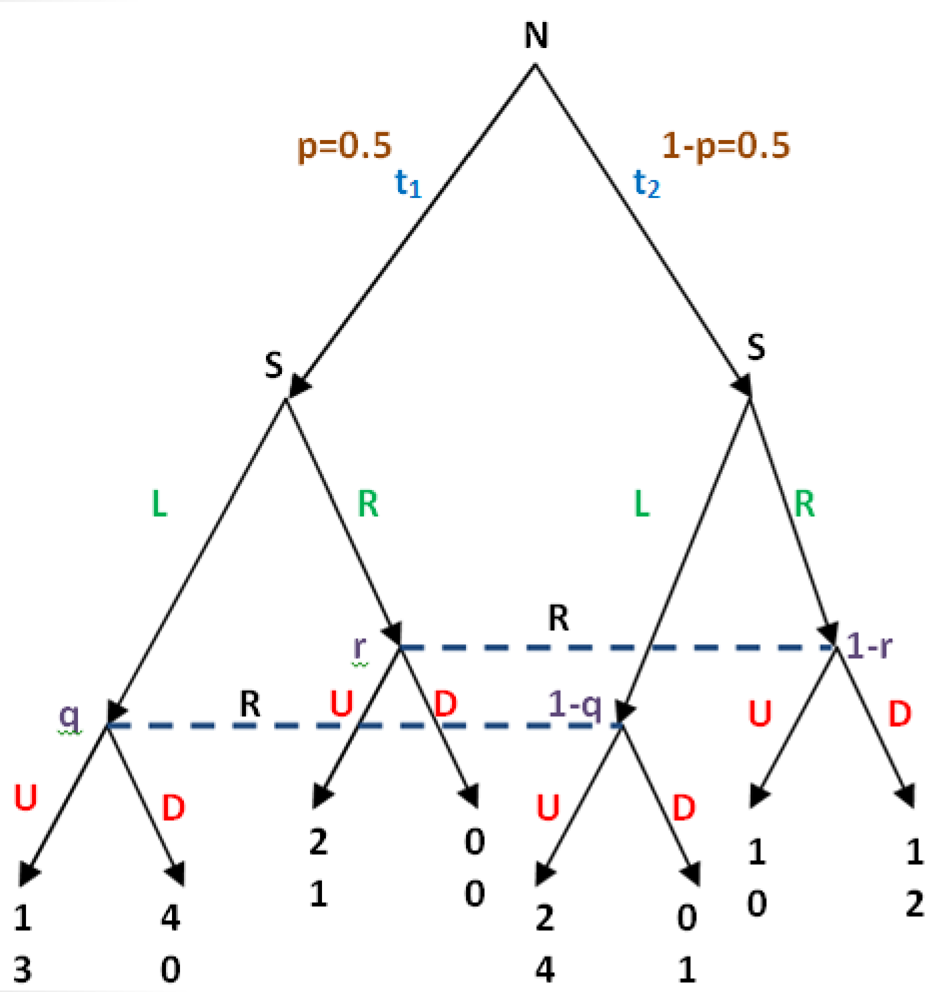
$$\mu(p|m_1) = \frac{Pr(p \cap m_1)}{Pr(m_1)} = \frac{pq}{pq + (1-p)(1-q)}$$

**Notice that there are no off-equilibrium paths. Thus, Requirement 4 is redundant.*

Definition: A pure strategy Perfect Bayesian Equilibrium in signaling games is a pair of strategies $m^*(t_i)$ and $a^*(m_j)$ and a belief $\mu(t_i|m_j)$ satisfying signaling requirements 1-3.

Perfect Bayesian Equilibrium in Signaling Games

Example



We will be looking for possible equilibria of two distinct types:

1. **Pooling equilibria**, i.e., the two types of the Sender are sending the same message.
2. **Separating equilibria**, i.e., the two types of the Sender are sending distinct messages