GAME THEORY

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LECTURE NOTES SET 7: AUCTIONS

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Auctions

- An auction, a major application of Bayesian games, is a mechanism for trading items by means of bidding.
- Auctions are a common method of allocating scarce goods across individuals with different valuations for these goods.
  - This corresponds to a situation of incomplete information because the valuations of different potential buyers are unknown.
    - For example, if we were to announce a mechanism, which involves giving a particular good (for example a seat in a sports game) for free to the individual with the highest valuation, this would create an incentive for all individuals to overstate their valuations.
- In general, auctions are designed by profit-maximizing entities, which would like to sell the goods to raise the highest possible revenue.

Auction formats

1. Sealed-bid auctions

- **First-Price Sealed-Bid Auction**: Everyone writes down a bid in secret. The person with the highest bid wins the object and pays what he bids.
- **Second-Price Sealed-Bid (Vickrey) Auction**: Everyone writes down a bid in secret. The person with the highest bid wins the object and pays the second highest bid.
- **All Pay Auction**: Everyone writes down a bid in secret. The person with the highest bid wins. Everyone pays.

2. Open-bid auctions

- **English Auction**: The auctioneer starts at a reserve price and increases the price until only one bidder is left.
- **Dutch Auction**: The auctioneer starts at a high price and decreases the price until a bidder accepts the price.

❖ *In this module we will discuss only sealed-bid auctions. All-pay auctions will be discussed during student presentations.*

Auctions and information structure

Auctions differ with respect to the information structure about the valuations of the bidders

1. **Private value auctions**
   - each bidder knows only her own value
   - examples: artwork, antiques, memorabilia
2. **Common value auctions**
   - actual value of the object is the same for everyone
   - bidders have different private information about that value
   - examples: oil field auctions, company takeovers

❖ *In this module we will discuss only private value auctions.*

Auctions modeling

- There is a single indivisible object is for sale.
- There is a fixed number of bidders, $\boldsymbol{i = 1, 2, \ldots n}$. In the analysis that follows we assume only two bidders.
- The bidders submit sealed bids. So bids are simultaneous and independent. The bid of bidder $\boldsymbol{i}$ is denoted $\boldsymbol{b_i}$
- Each bidder $\boldsymbol{i}$ has a valuation for the object, denoted by $\boldsymbol{v_i}$. He enjoys $\boldsymbol{v_i}$ if and only if he wins the object.
- The highest bidder wins the object. If there are two or more bids tied at the top, the winner is chosen randomly (where all top bidders win with positive probability).
- The price paid depends on the type of the auction (first-price sealed-bid, or second-price sealed-bid).
- Only the winning bidder pays. The price paid for the object is denoted by $\boldsymbol{p}$.
- Bidders are risk-neutral. If $\boldsymbol{i}$ wins the object and the price is $\boldsymbol{p}$, his utility is $\boldsymbol{v_i - p}$. All non-winners have a utility of 0.
- If there is uncertainty about who wins, expected payoffs guide the bidders' behavior

First-price sealed-bid auction with private values

- We consider the case where each bidder $i = 1, 2$ knows v_i but is uncertain about the valuation of the other bidder.
  - So, it is as if “Nature” assigns a valuation to each bidder. This defines a game of incomplete information.
  - Nature draws a vector (v_1, v_2) according to the uniform probability distribution.
- It is logical to assume that each bidder will bid a percentage of her own valuation no more than 100% (why?)
  - Therefore, $b_i = kv_i$ with $k \in [0, 1]$
- What will the expected payoff of player 1 be?
  - It is

$$EU_1 = \begin{cases} v_1 - b_1 & if\ b_1 > b_2 \\ r(v_1 - b_1) & if\ b_1 = b_2 \\ 0 & if\ b_1 < b_2 \end{cases}$$

First-price sealed-bid auction with private values

Notes:

- The probability that two values drawn from a continuous probability distribution are equal is zero.
- The probability that a number drawn from a continuous probability distribution is less than a specific value is represented the cumulative probability function up to that specific value.
- Hence the expected payoff of player 1 is

$$\begin{aligned} \boldsymbol{EU_1} &= \boldsymbol{Prob(b_1 > b_2)(v_1 - b_1)} + \\ &+ \boldsymbol{Prob(b_1 = b_2) r (v_1 - b_1)} + \\ &+ \boldsymbol{Prob(b_1 < b_2) 0} \Rightarrow \end{aligned}$$

$$\boldsymbol{EU_1 = Prob(b_1 > b_2)(v_1 - b_1)}$$

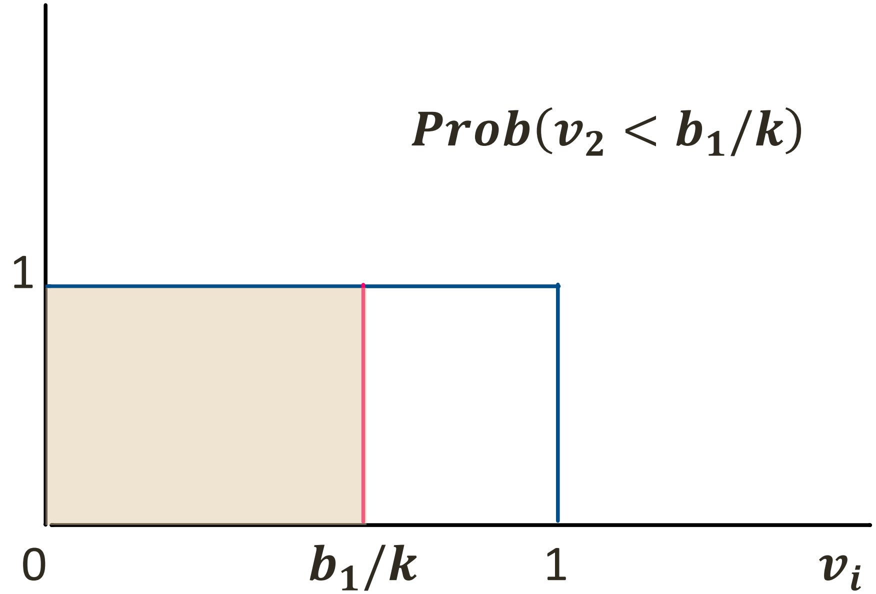
- What is $\boldsymbol{Prob(b_1 > b_2)}$?

First-price sealed-bid auction with private values

- What is $\boldsymbol{Prob(b_1 > b_2)}$?

  Remember that players bid a percentage up to 100% of their valuation. Therefore, player 1 believes that player 2 will bid according to $\boldsymbol{b_2 = kv_2}$. Hence, $\boldsymbol{Prob(b_1 > b_2) = Prob(b_1 > kv_2) = Prob(v_2 < b_1/k)}$

  Graphically, when considering the uniform, distribution it will be equal to the shaded area below:



Hence, $\boldsymbol{Prob(v_2 < b_1/k) = 1 \times b_1/k}$

First-price sealed-bid auction with private values

Therefore, bidder 1 is picking her bid in order to maximize

$$EU_1 = Prob(b_1 > b_2)(v_1 - b_1) = (b_1/k)(v_1 - b_1)$$

The first order condition of the above objective function is

$$\frac{\partial EU_1}{\partial b_1} = 0 \Rightarrow \frac{v_1}{k} - \frac{2b_1}{k} = 0 \Rightarrow b_1^* = v_1/2$$

So, the BNE of this “game” is

$$BNE = \{b_1^* = \frac{v_1}{2}, b_2^* = \frac{v_2}{2}, v_i \sim U[0,1] \ \forall i = 1,2\}$$

First-price sealed-bid auction with private values

What can we say about first-price sealed-bid auctions with private values?

- This auction is efficient.
  - The bidder with the highest valuation will get the item, hence the highest possible value added is generated through this auction.
- The bidders will bid less than their valuations.
  - Every bidder wants to win → incentive to raise bid
  - Conditional on winning, a bidder wants to pay as little as possible → incentive to lower bid
  - Hence, a bidder trades off the probability of winning with the possibility of making higher profits when she wins.

Second-price sealed-bid auction with private values

- We consider the case where each bidder $\boldsymbol{i = 1, 2}$ knows $\boldsymbol{v_i}$ but is uncertain about the valuation of the other bidder.
  - So, it is as if "Nature" assigns a valuation to each bidder. This defines a game of incomplete information.
  - Nature draws a vector $(\boldsymbol{v_1, v_2})$ according to the uniform probability distribution.
- What will the expected payoff of player 1 be?
  - It is

$$\boldsymbol{EU_1} = \begin{cases} \boldsymbol{v_1 - b_2} & \boldsymbol{if\ b_1 > b_2} \\ \boldsymbol{r(v_1 - b_2)} & \boldsymbol{if\ b_1 = b_2} \\ \boldsymbol{0} & \boldsymbol{if\ b_1 < b_2} \end{cases}$$

Second-price sealed-bid auction with private values

Working as in the case of first-price we can show that bidder 1 is picking her bid in order to maximize

$$\boldsymbol{EU_1 = Prob(b_1 > b_2)(v_1 - b_2) = (b_1/k)(v_1 - b_2)}$$

Note:

- This objective function is linear on $\boldsymbol{b_1}$
- The profit of a bidder if she wins does not change with her own bid: she can increase the probability of winning without sacrificing profit!

The above imply that

- Bidding one's value is a (weakly) dominant strategy independent of the "type" (i.e., the valuation) of the rival

So, the BNE of this "game" is

$$\boldsymbol{BNE = \{b_1^* = v_1, b_2^* = v_2, v_i \sim U[0,1]\ \forall i = 1,2\}}$$

Second-price sealed-bid auction with private values

What can we say about second-price sealed-bid auctions with private values?

- This auction is efficient.
  - The bidder with the highest valuation will get the item, hence the highest possible value added is generated through this auction.
- The bidders will bid their valuations.
  - Every bidder wants to win → incentive to raise bid
  - Conditional on winning, a bidder wants to pay as little as possible → the payment does not depend on their own bid!
  - Hence, a bidder does not trade off the probability of winning with the possibility of making higher profits when she wins.

First-price or second-price

- Both types of auction are efficient.
  - The bidder with the highest valuation will get the item, hence the highest possible value added is generated through this auction.
- Which one generates the highest revenue for the seller?
  - In the first-price bidders bid below their valuations → the revenue for the seller is below the valuation of the highest bidder
  - In the second-price bidders bid their valuations but the winner pays the highest valuation among the rest of the bidders → the revenue for the seller is below the valuation of the highest bidder

Is there something we can say? Yes!

Revenue Equivalence Theorem

Any auction with independent private values and a common distribution where

1. the number of the bidders are the same and the bidders are risk-neutral,
2. the object always goes to the buyer with the highest value,
3. the bidder with the lowest value expects zero surplus,

yields the same expected revenue