GAME THEORY

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LECTURE NOTES SET 6: INFORMATION AND GAMES – STATIC BAYESIAN GAMES

Eleftherios Filippiadis

Office: ΓΔ3, 310
Phone number: 2310-891770
Email: efilipp@uom.gr

Information in Games

Perfect information → each player, every time she is asked to make a decision, is perfectly informed about the previous actions of all other players.

Complete information → the structure of the game is common knowledge (i.e., the order of decision making, utility functions, payoffs, strategies and "types" of players)

Hence, there are games of

- perfect and complete information (e.g., the extensive form games we have covered so far)
- imperfect but complete information (e.g., the static form games we have covered so far)
- incomplete information w/ or w/o perfect information

Information in Games

What if the information is incomplete and/or imperfect?

The solution concept of Nash Equilibrium for the static games (of compete information) consider one action/strategy per player…

…

but in static games with incomplete information players have …multiple personalities!

⇓

Nash Equilibrium is not appropriate for these games!

What should we do?

- We must introduce a “device” that describes the information available to each and every player → Baye’s rule and Common Priors
- We must redefine strategies → one strategy per type!
- We must “refine” the solution concepts accordingly

Information sets

Information sets

Definition: Information set I_i of player $i \in \mathcal{N}$ is a set of nodes such that

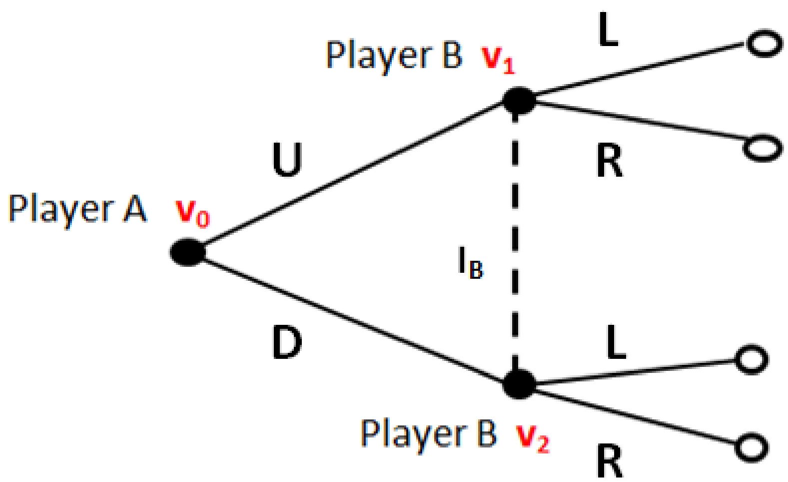
- $I_i \subset X_i$, that is if I_i belongs to player i, so do the nodes in I_i
- $\forall v_i, v_i{}' \in X_i$ we get $A_{v_i} = A_{v_i{}'}$, that is if any two nodes belong to the same information set then they have the same actions
- $\forall v_i, v_i{}' \in X_i$ one cannot be a predecessor of the other

Strategies

- **Definition:** a **strategy** s_i of player $i \in \mathcal{N}$ is a mapping from the set of information to the set of actions, i.e., s_i: $I_i \rightarrow A_{X_i}$

Information sets

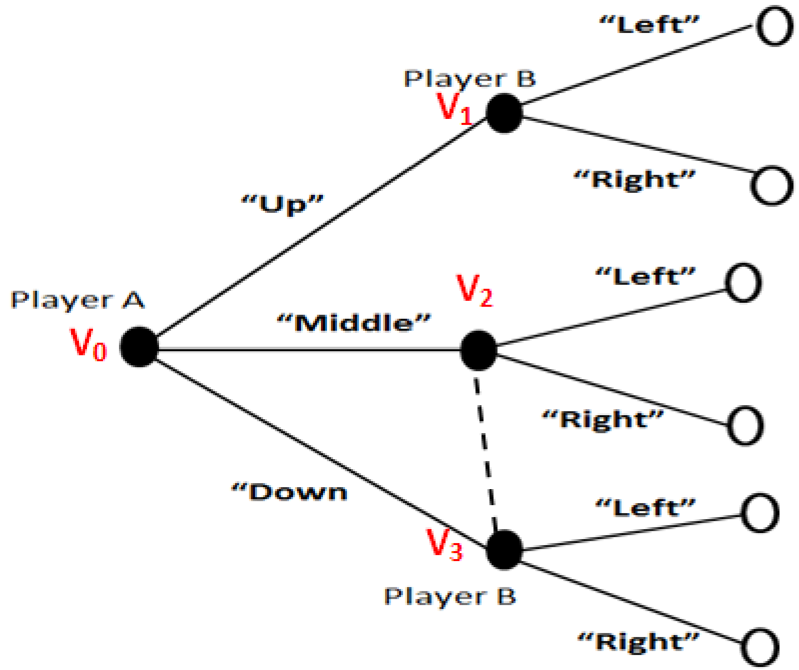
Example 1



- the information set of player A includes only node v_0 (a singleton).
- the information set of player B includes the nodes v_1 and v_2.
- the actions of player B in each of the nodes of his information set are identical (otherwise, player B can infer the node he is in by the actions available to him).
- the possible strategies of the two players are $\boldsymbol{s_A = \{U, D\}}$ and $\boldsymbol{s_B = \{L, R\}}$.

Information sets

Example 2



- the information set of player A includes only node v_0 (a singleton).
- the information sets of player B are $I_{B1}=\{v_1\}$ and $I_{B2}=\{v_2, v_3\}$
- the actions of player B in each of the nodes of his information set are identical (otherwise, player B can infer the node he is in by the actions available to him).
- the possible strategies of the players are $S_A = \{U, M, D\}$ and $S_B = \{OL, OR, IL, IR\}$.

Information sets

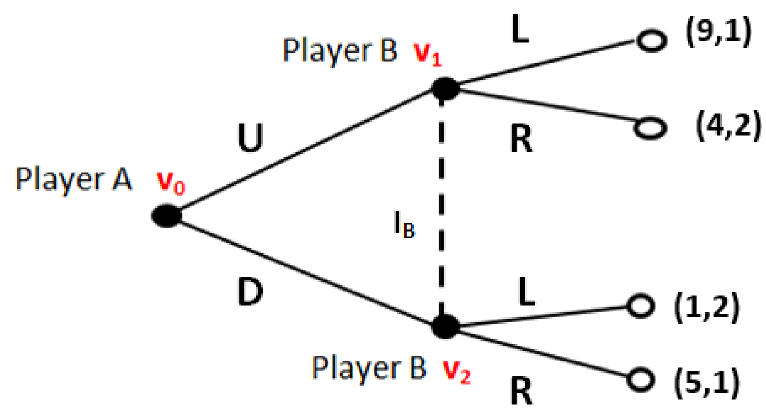
Notes:

- we cannot have any terminal nodes in an information set
- we are assuming "perfect recall", i.e., any two nodes in an information set of a specific player originate from the same action that specific player has taken in the past.

Converting a Normal Form Game into a Game Tree

		Player B	
		Left	Right
Player A	Up	9, 1	4, 2
	Down	1, 2	5, 1

$\Leftrightarrow$



Note: in games like the one above, we cannot find the SPNE using backward induction. Instead, we are looking for the NE that coincide with the SPNE.

Static Games with Incomplete Information

We consider now situations where player i might be unsure about the payoffs of another player. For player i to continue it is necessary that she forms beliefs about the other player. These beliefs should reflect all available information player i has.

ELEMENTS OF A STATIC GAME OF …

… Complete Information	… Incomplete Information

• Players	• Players
	  • Types
	  • Common prior beliefs

• Actions	• Actions
• Payoffs	• Payoffs

Static Games with Incomplete Information

Bayesian Game (a description)

Each player can have different types $\boldsymbol{t_i} \in \boldsymbol{T_i}$, where $\boldsymbol{T_i}$ is the collection of all possible types of player $\boldsymbol{i}$. A type for each player, that is $\boldsymbol{t} = (\boldsymbol{t_1}, \boldsymbol{t_2}, \dots, \boldsymbol{t_n}) \in \boldsymbol{T} = \times_{i \in N} \boldsymbol{T_i}$, is randomly drawn with probability $\boldsymbol{p(t)}$ (by "Nature"). The probability distribution over all possible combinations of types is known by all players at the beginning of the game (common priors). Then, each player learns his type but not the others' and they update their beliefs, when possible, based on Baye's Rule

$$\boldsymbol{p(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)}}$$

Then, the players simultaneously choose their actions. The set of actions chosen is denoted $\boldsymbol{a} = (\boldsymbol{a_1}, \boldsymbol{a_2}, \dots, \boldsymbol{a_n}) \in \boldsymbol{A} = \times_{i \in N} \boldsymbol{A_i}$. Hence, the payoffs of any player $\boldsymbol{i}$ will depend on players' types and actions, i.e., $\boldsymbol{u_i}: \boldsymbol{A} \times \boldsymbol{T} \to \boldsymbol{\mathcal{R}}$. Such a game can be stated as $\langle \boldsymbol{\mathcal{N}}, \{\boldsymbol{T_i}\}_{i \in \mathcal{N}}, \{\boldsymbol{A_i}\}_{i \in \mathcal{N}}, \boldsymbol{p}, \{\boldsymbol{u_i}\}_{i \in \mathcal{N}} \rangle$ and is called a ***Bayesian Game***.

Static Games with Incomplete Information

Bayesian Nash Equilibrium (BNE)

- A BNE is a set of strategies, one for each type of every player, such that no type has inventive to change her strategy given the beliefs about the types and the strategies of all other types of players.
- The beliefs are based on the initial probability distribution over the set of types (priors) and they must be updated, when possible, according to Bayes' Rule

$$\boldsymbol{p(A|B) = \frac{p(A \cap B)}{p(B)}}$$

Definition: A strategy profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is a Bayesian Nash Equilibrium in an n-person static game of incomplete information if and only if for each player $\boldsymbol{i} \in \boldsymbol{\mathcal{N}}$ and type $\boldsymbol{t_i} \in \boldsymbol{T_i}$

$$\boldsymbol{s_i^*(t_i) \in \arg\max_{a_i} \sum \mathrm{u_i}(s_1^*(t_1), \dots, a_i, \dots, s_n^*(t_n)) \times p_i(t'_{-i}|t_i)}$$

Static Games with Incomplete Information

Example 1

Consider a static Bayesian game with two players where Nature decides whether player 1 is type A with probability $\boldsymbol{p}$ or type B with probability $(\mathbf{1}-\boldsymbol{p})$. The corresponding payoffs are as in payoff matrices A and B (see below). Player 1 is informed about her "type" while player 2 only knows the probability distribution over the types.

		Player 2	
		Left	Right
Player 1^A	Up	1, 1	0, 0
	Down	0, 0	2, 2

		Player 2	
		Left	Right
Player 1^B	Up	2, 1	0, 2
	Down	0, 0	1, 2

What are the Bayesian Nash Equilibria of this game?

Static Games with Incomplete Information

Example 2

Let a Cournot duopoly model with inverse demand $P = 10 - Q$, where $Q = q_1 + q_2$ is the aggregate quantity on the market. Firm 2's cost function is $C_2(q_2) = 2q_2$. Firm 1's cost function is

$$C_1 = \begin{cases} 2q_1, & \text{with probability } p \\ q_1 & \text{with probability } 1 - p \end{cases}$$

Furthermore, information is asymmetric: firm 1 knows its cost function and firm 2's, but firm 2 knows its cost function and only that firm 1's marginal cost is 2 with probability p and 1 with probability $1 - p$. All of this is common knowledge: Firm 1 knows that firm 2 has superior information, firm 2 knows that firm 1 knows this, and so on.

What is the Bayesian Nash Equilibrium of this example?