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Homework on Sequential Games

This fourth homework is due in class next week (7/3) at the start of our lecture. You can work in groups of up to 3 students and turn in a single homework for your group.

**Exercise 1:** Final (2017) Armies A and B are fighting over an island initially held by army B. The only way to capture the island is launching a missile on the occupying army. Initially, army A is endowed with 2 missiles and army B is endowed with 1 missile. In each period the army that does not occupy the island can launch an attack if it has any missiles left. As the result of such an attack, the attacking army loses one missile, incurs the costs of the launch, and occupies the island. If the non-occupying army decides not to attack in a given period, the war ends.

Let $n_i$ denote the number of missiles army $i$ has launched during the war and let $x = 1$ if army A occupies the island at the end of the war, and $x = 0$ if army B does. The commander of army A's payoff function is $u_A(n_A, x) = 2x - n_A$, and the commander of army B's payoff function is $u_B(n_B, x) = 2(1 - x) - n_B$.

(5) a) How many (pure) strategies does each of the players have in this game?

(10) b) Find the *subgame perfect equilibria* of this game.

**Exercise 2:** (Final, 2018) Two friends, Sam (she) and Jan (he), must decide independently where to meet after class. The three possible choices are a bar called Foy's, the local mall Trois Fountaines, or the restaurant called l'Atelier. Sam and Jan have preferences over these three spots, but they also have a general desire to be together, rather than apart. More specifically,

- Sam's first choice is to be with Jan at the Foy's, second is to be with Jan at the Trois Fountaines, third is to be alone at the Foy's, fourth is to be with Jan in l'Atelier, fifth is to be at the Trois Fountain alone, and the last is to be alone in l'Atelier.

- Jan's ranking is, from best to worst, be with Sam at l'Atelier, be with Sam at Trois Fountaines, be with Sam at the Foy's, be alone at Trois Fountaines, be alone at l'Atelier, be alone at the Foy's.

To complete the preferences, we assume that, if they could not coordinate on going to the same place, i.e., if the other person is somewhere else, it does not matter to Sam or Jan where is that somewhere else. For example, if Sam is at the Foy's and Jan is not there, it does not matter for Sam if Jan is at Trois Fountaines or at l'Atelier.

a) Suppose that Sam and Jan must choose independently where to go, without knowing what the other party has done. Represent the normal form for this game with a payoff matrix. Which strategies survive the iterated elimination of strictly dominated strategies? What are the pure-strategy Nash equilibria?

b) Now suppose instead that Jan moves first: Jan chooses a location among the three, goes there, and phones Sam, saying reliably and credibly, "I am at location X, and I am not moving." After this, Sam decides where to go. represent the extensive form of this game with a game tree. How many strategies does each of the players have? Find the subgame-perfect Nash equilibria of this game in pure strategies.

c) Find a Nash equilibrium which is not subgame perfect.

**Exercise 3:** (Midterm 2019, 25 points) Two players must choose among three alternatives, a, b, and c. Player 1 prefers a to b to c, while Player 2 prefers b to a to c. The rules are that player 1 moves first and can veto one of the three alternatives. After observing Player 1's veto, Player 2 chooses one of the remaining two alternatives.

(5) a) Model this as an extensive-form game tree (choose payoffs that represent preferences).

(5) b) How many pure strategies does each player have?

(10) c) Find the Subgame Perfect Nash Equilibrium of this game.

(5) d) Find a Nash Equilibrium of this game which is not subgame perfect.

**Exercise 4:** (Final 2019, 30 points) *Multiple equilibria as a cooperation device.* Adam and Eve are living in the same cave. The first thing that each of them does in the morning is to decide whether to contribute to the cleaning of the cave. The decisions are made simultaneously and the payoffs are given by the following matrix:

| | | Eve | |
|---|---|---|---|
| | | clean up | mess up |
| Adam | clean up | $2,2$ | $-2,3$ |
| | mess up | $3,-2$ | $1,1$ |

After they make their cleaning/messing up decisions and observing the decision of the other partner, now each of them decides whether to go hunting a stag or hunting a hare. The payoffs from this part of their interaction are given by:

| | | Eve | |
|---|---|---|---|
| | | stag | hare |
| Adam | stag | $5,5$ | $0,3$ |
| | hare | $3,0$ | $3,3$ |

We will consider the sequential game where Adam and Eve first make choices in the first matrix, then they observe each others' choices, and finally they make their choices in the second matrix. Each player's aim is to maximize the payoff he/she gets from the first matrix plus the payoff he/she gets from the second matrix.

(7) a) How many subgames does this sequential game have?

(8) b) Since this is a symmetric game, both players have the same number of pure strategies. Find this number.

(15) c) There are multiple subgame-perfect Nash Equilibria of this game. Find one of the subgame-perfect Nash Equilibria that maximizes the sum of the player payoffs.

**Exercise 5**: (Final 2020, 20 points) The output of a firm is $L\,(40 - L)$ as a function of its laborforce $L$. The per unit price of its product is 1. A union representing the workers decides on the wage level $w$, and after observing this wage level the firm decides on the magnitude of the laborforce $L$. The union maximizes the wage payments ($wL$) and the firm maximizes its profits (value of its output minus the wage payments). Assume that $L$ and $w$ cannot be set higher than 40.

(5) a) Find the optimal laborforce employment decision $L$ of the firm as a function of the wage level $w$.

(5) b) Backward induction: Given the profit-maximizing laborforce decision of the firm, what is the optimal wage level that the union should set?

(5) c) Find the *subgame-perfect Nash equilibrium* of this game.

(5) d) Is the equilibrium outcome you found above *Pareto efficient*? Why or why not?