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Homework on Sequential Games
Sketch of the Suggested Solutions

Exercise 1: Final (2017) Armies A and B are fighting over an island initially held by army B. The only way to capture the island is launching a missile on the occupying army. Initially, army A is endowed with 2 missiles and army B is endowed with 1 missile. In each period the army that does not occupy the island can launch an attack if it has any missiles left. As the result of such an attack, the attacking army loses one missile, incurs the costs of the launch, and occupies the island. If the non-occupying army decides not to attack in a given period, the war ends.

Let n_i denote the number of missiles army i has launched during the war and let $x = 1$ if army A occupies the island at the end of the war, and $x = 0$ if army B does. The commander of army A's payoff function is $u_A(n_A, x) = 2x - n_A$, and the commander of army B's payoff function is $u_B(n_B, x) = 2(1-x) - n_B$.

(5) a) How many (pure) strategies does each of the players have in this game?

When we draw the game tree, we see that player A is moving on two decision nodes (to choose whether to attack or not) and player B is moving on one node. The number of pure strategies for player 1 is 2x2=4, and the number of pure strategies for player B is 2.

(10) b) Find the *subgame perfect equilibria* of this game.

Player A: attack, attack; Player B: do not attack

Exercise 2: (Final, 2018) Two friends, Sam (she) and Jan (he), must decide independently where to meet after class. The three possible choices are a bar called Foy's, the local mall Trois Fountaines, or the restaurant called l'Atelier. Sam and Jan have preferences over these three spots, but they also have a general desire to be together, rather than apart. More specifically,

- Sam's first choice is to be with Jan at the Foy's, second is to be with Jan at the Trois Fountaines, third is to be alone at the Foy's, fourth is to be with Jan in l'Atelier, fifth is to be at the Trois Fountain alone, and the last is to be alone in l'Atelier.

- Jan's ranking is, from best to worst, be with Sam at l'Atelier, be with Sam at Trois Fountaines, be with Sam at the Foy's, be alone at Trois Fountaines, be alone at l'Atelier, be alone at the Foy's.

To complete the preferences, we assume that, if they could not coordinate on going to the same place, i.e., if the other person is somewhere else, it does not matter to Sam or Jan where is that somewhere else. For example, if Sam is at the Foy's and Jan is not there, it does not matter for Sam if Jan is at Trois Fountaines or at l'Atelier.

a) Suppose that Sam and Jan must choose independently where to go, without knowing what the other party has done. Represent the normal form for this game with a payoff matrix. Which strategies survive the iterated elimination of strictly dominated strategies? What are the pure-strategy Nash equilibria?

Answer: The example is taken from the "Microeconomics for Managers" textbook of David M. Kreps. This is his lead example for the chapter on non-cooperative game theory. The exact payoffs are not important. What is important is they satisfy the ordinal ranking for each player.

Jan

Sam

	$Foy's$	$3Fount$	$l\prime Atelier$
$Foy's$	$6, 4$	$4, 3$	$4, 2$
$3Fountaines$	$2, 1$	$5, 5$	$2, 2$
$l'Atelier$	$1, 1$	$1, 3$	$3, 6$

For Sam, strategy $Foy's$ strictly dominates strategy $l'Atelier$. Once Sam's strategy $l'Atelier$ is deleted, in the reduced game, strategy $3Fount$ strictly dominates strategy $l\prime Atelier$ for Jan. In the remaining game 2x2 game, no other strategy is strictly dominated. In conclusion, strategies $Foy's$ and $3Fount$ both survive iterated elimination of strictly dominated strategies for both players. After identifying the best response functions, we can also see that there are two pure-strategy Nash equilibria of this game: i) both players choose to go to $Foy's$; ii) both players choose to go to $3Fount$.

b) Now suppose instead that Jan moves first: Jan chooses a location among the three, goes there, and phones Sam, saying reliably and credibly, "I am at location X, and I am not moving." After this, Sam decides where to go. represent the extensive form of this game with a game tree. How many strategies does each of the players have? Find the subgame-perfect Nash equilibria of this game in pure strategies.

Answer: Jan has to choose between the three locations, therefore he has three strategies. Sam observes Jan's choice and than decides where to go. In other words, she has three different decision nodes in the game. On each of these nodes, there are three possible actions. So Sam has $3 \times 3 \times 3 = 27$ strategies. By using backward induction on the preferences of Sam, we can identify the SPNE strategy of Sam: Go to the Foy's if Jan is at the Foy's, Go to Trois Fountaines if Jan is at Trois Fountaines, Go to Foy's if Jan is at l'Atelier. Given Sam's sequentially rational strategy, Jan is deciding among three options: Foy's with Sam, Trois Fountaines with Sam, or l'Atelier alone. Considering Jan's preferences, he would choose going to the Trois Fountaines.

c) Find a Nash equilibrium which is not subgame perfect.

Answer: This is a rather large game. I will not write down the entire payoff matrix since it requires a 3×27 matrix. Instead, I will try to identify a non-subgame-perfect Nash equilibrium directly. Such an equilibrium should be based on a non-credible threat that Sam makes, such as "I will go to $Foy's$ no *matter where you go." Jan's best response to such a non-sequentially-rational strategy is going to Foy's as well. Notice that if these are the strategies of the two players, none of them has a profitable deviation at the start of the game.*

Exercise 3: (Midterm 2019, 25 points) Two players must choose among three alternatives, a, b, and c. Player 1 prefers a to b to c, while Player 2 prefers b to a to c. The rules are that player 1 moves first and can veto one of the three alternatives. After observing Player 1's veto, Player 2 chooses one of the remaining two alternatives.

(5) a) Model this as an extensive-form game tree (choose payoffs that represent preferences).

The game will have two stages. Player1 will move on the first decision node and choose one of the three branches of the game tree: veto a, veto b, or veto c. This gives us three more decision nodes on which player 2 will act. For each of these decision nodes, player 2 will choose one of the two branches. For instance, if a is vetoed already, the options available to player 2 are b and c. If b is vetoed, they are a and c. If c is vetoed, they are a and b. This gives us a total of 6 different ways that the game can be played by the players, yielding the 6 terminal nodes of the game. The payoffs can be chosen according to the preferences of the players given in the question.

(5) b) How many pure strategies does each player have?

Player 1 has 3 pure strategies (veto a, veto b, and veto c). Player 2's strategy should tell her what to do on each of the 3 decision nodes that she plays. Accordingly Player 2 has $2 \times 2 \times 2 = 8$ *strategies.*

(10) c) Find the Subgame Perfect Nash Equilibrium of this game.

By using backward induction, sequential rationality implies that player 2 chooses b when a is vetoed by player 1, chooses a when b is vetoed, chooses b when c is vetoed. Going backwards to the root of the game tree, we see that player 1 would veto alternative b. The subgame perfect Nash equilibrium is player 1 vetoes b, player 2 chooses strategy (b,a,b). Player 1's behavior is an example to "strategic voting." Even though this player prefers b to c, he ends up vetoing b instead of c, because this is the only way to ensure that his favorite option of a will be chosen by the other player.

(5) d) Find a Nash Equilibrium of this game which is not subgame perfect.

You can find all the Nash equilibria by writing down the 3×8 payoff matrix. I will take a short-cut and construct one where alternative b is chosen. Notice that this is the favorite alternative of player 2, so there will be a Nash equilibrium based on non-credible threats: Suppose player 2 chooses alternative c unless c is vetoed by player 1. Otherwise, if c is vetoed, player 2 chooses b. In other words, player 2's strategy is (c,c,b). Notice that, player 1's best response to (c,c,b) is vetoing alternative c (this is the only way player 1 can avoid c). Player 2's strategy would be a best response to a veto on c, as long as she chooses b after this veto (instead of a). So we have two strategies which are best response to each *other. Accordingly, player 1 vetoes c, player 2 chooses strategy (c,c,b) is a Nash equilibrium which is not subgame perfect.*

Exercise 4: (Final 2019, 30 points) *Multiple equilibria as a cooperation device.* Adam and Eve are living in the same cave. The first thing that each of them does in the morning is to decide whether to contribute to the cleaning of the cave. The decisions are made simultaneously and the payoffs are given by the following matrix:

		Eve	
		clean up	mess up
Adam	clean up	$2, 2$	$-2, 3$
	mess up	$3, -2$	$1, 1$

After they make their cleaning/messing up decisions and observing the decision of the other partner, now each of them decides whether to go hunting a stag or hunting a hare. The payoffs from this part of their interaction are given by:

		Eve	
		stag	hare
Adam	stag	$5, 5$	$0, 3$
	hare	$3, 0$	$3, 3$

We will consider the sequential game where Adam and Eve first make choices in the first matrix, then they observe each others' choices, and finally they make their choices in the second matrix. Each player's aim is to maximize the payoff he/she gets from the first matrix plus the payoff he/she gets from the second matrix.

(7) a) How many subgames does this sequential game have?

The first stage of the game can end in 4 different ways. At each of these 4 nodes, a stag hunt subgame starts. 4 subgames + the entire game itself gives us the number 5 as the number of subgames of the game.

(8) b) Since this is a symmetric game, both players have the same number of pure strategies. Find this number.

Adam will choose between two actions in the first stage. He also has two actions in each of the 4 stag hunt subgames. So he has

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 \text{ strategies}$$

It was not part of the question, but the extensive form of this game can be represented with the following game tree (attached).

(15) c) There are multiple subgame-perfect Nash Equilibria of this game. Find one of the subgame-perfect Nash Equilibria that maximizes the sum of the player payoffs.

We find the SPNE by backward induction. The stag-hunt subgames have two Nash equilibria in pure strategies: Either (stag,stag) or (hare,hare). Since we are asked to find the payoff-maximizing SPNE, an immediate reaction to this question could be thinking that the players will play (stag,stag) in the second stage. But this would be an incomplete description of the equilibrium in the second stage of the entire game, since it does not give the complete strategies in the second stage. Since there are 4 different subgames, we have to specify an equilibrium for each of these 4 subgames. What about (stag,stag) in all the 4 stag-hunt subgames. Then going backwards, we see that Adam and Eve would play (mess up, mess up) in the first stage, since the first stage is the re-incarnation of the prisoners' dilemma. This is indeed a SPNE. The equilibrium payoffs are 1+5=6.

There are many other SPNE which give the payoff 6 to the players. For instance, they can play (stag,stag) only in the stag-hunt subgame following (mess up, mess up) in the first stage, but play (hare,hare) in the other 3 stag-hunt subgames.

But the fact that the players may play different Nash equilibria in different subgames should give us the idea that these different equilibria can be used as a commitment device to cooperate in the first stage of the game as well. Suppose Adam and Eve are playing (stag,stag) equilibrium only in the subgame following (clean up, clean up) in the first stage, and they play (hare,hare) in the remaining 3 stag-hunt subgames. Now, you should notice that you can support (clean up, clean up) as part of the SPNE behavior in the first stage: Suppose you know the other player will clean up. Of course, you can choose to mess up and it will give you an extra payoff of 3-2=1 in the first stage. But this also guarantees that you play (hare,hare) in the second stage of the game, costing you a reduction of 5-3=2 in your payoff.

In conclusion, there exists a SPNE of this sequential game where each player gets the payoff 2+5=7. This exercise is inspired by Roger Myerson's article "Learning from Schelling's Strategy of Conflict" Journal of Economic Literature, 2009.

Exercise 5: (Final 2020, 20 points) The output of a firm is $L(40-L)$ as a function of its laborforce L. The per unit price of its product is 1. A union representing the workers decides on the wage level w, and after observing this wage level the firm decides on the magnitude of the laborforce L. The union maximizes the wage payments (wL) and the firm maximizes its profits (value of its output minus the wage payments). Assume that L and w cannot be set higher than 40.

(5) a) Find the optimal laborforce employment decision L of the firm as a function of the wage level w.

$$\max_{L} L(40-L) - wL$$

From first-order conditions, we see that $40 - w - 2L = 0$*, implying that the profit maximizing laborforce level is* $L(w) = \frac{40-w}{2}$.

(5) b) Backward induction: Given the profit-maximizing laborforce decision of the firm, what is the optimal wage level that the union should set?

$$\max_w wL(w) = \max_w w\frac{40-w}{2}$$

From the first-order conditions, $\frac{40-2w}{2} = 0$*, implying that the optimal wage rate for the union is* $w = 20$.

(5) c) Find the *subgame-perfect Nash equilibrium* of this game.

Here are the SPNE strategies: $w = 20$ *for the union, and* $L(w) = \frac{40-w}{2}$ *for the firm.*

(5) d) Is the equilibrium outcome you found above *Pareto efficient*? Why or why not?

Under the SPNE strategies, $w = 20$ *and* $L(w) = \frac{40-w}{2} = 10$*. Notice that this laborforce level does not maximize the magnitude of the pie to be shared by the two players* $L(40-L)$*. So the equilibrium outcome is not Pareto efficient.*

To be more concrete, the payoffs under the equilibrium outcome are $wL = 20 \times 10 = 200$ *for the union and* $L(40-L) - wL = 10 \times 30 - 20 \times 10 = 100$ *for the firm. With the alternative outcome* $w' = 10$ *and* $L' = 20$*, there is no change in the union's payoff, but the firm's payoff increases to* $20 \times 20 - 10 \times 20 = 200$.