GAME THEORY

SPRING 2022

LECTURE NOTES SET 1: INTRODUCTION

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What Is Game Theory* …?

- Game Theory is a methodological tool.
  - Philosophers, economists and other social scientists were getting their hands around it for centuries
  - Mathematically formulated for the first time by John von Neumann and Oscar Morgenstern (1944)
  - Widespread application of game theory in economics (among other fields) in the last 60 years
- It allows us to formally describe specific type of situations and "solve" them. It models **strategic situations**, that is, situations where each agent's action has consequences on other agents' well-being.
  - Agents might try to resolve the situation acting individually (non-cooperatively)
  - Agents might try to coordinate their actions to resolve the situation (cooperatively)

  @ We will only consider non-cooperative game theory in this class @
- In such situations traditional optimization fails.

**Game Theory is a misnomer for Multi-person Decision Theory*

Some Applications Of Game Theory

- Oligopolistic competition, where one firm's price or quantity choices affect another firm's profits through a common market demand.
- Auctions, where one bidder's bid affects another bidder's profit.
- Bilateral and multilateral bargaining.
- Externalities where, by definition, one agent's action affects another agent's well-being.
- Contract theory where a contract has to be designed in an incentive compatible fashion.
- Moral hazard problem occurs when the actions of one party may change to the detriment of another after a financial transaction has taken place (informational asymmetry, principal-agent problem).
- Market design (one-sided/two-sided, one-to-one, many-to-one matchings).
- Coalition formation (cartels, international environmental agreements, clubs, trading blocks, coalitional governments).
- Cost sharing/surplus division.

Representation Of Games

Strategic interactions (games) can be analyzed once we define

- all the **agents** (players) whose decisions must be taken into account (*be careful: there might be "others" whose actions must be taken into account but THEY are not players*)
- the **actions** and the **information** available to them
  - actions are the choices available to a player anytime she has to make a decision affecting the final outcome; a collection of a players actions, one every time she has to make a choice, is a strategy.
  - Information may be complete or incomplete, perfect or imperfect; we will properly describer them later.
- the "protocol" according to which players choose their actions in order to reach an **outcome**
- their **preferences** over all possible outcomes (*based on payoffs*)

There are two ways to represent situations of strategic interactions

1. strategic (or normal) form games, where players choose "simultaneously," and
2. extensive form games, where players choose their actions sequentially according to a specific order (i.e., a "protocol").

PART I: STRATEGIC (OR NORMAL FORM) GAMES OF COMPLETE INFORMATION

A strategic situation involves a number of agents where each agent must take at least one action while having specific preferences over the set of potential outcomes (payoffs). The payoff each agent receives at the end depends not only on her own actions but on the actions of all other players as well.

We first consider static games of complete information, that is

- players simultaneously choose actions (*static game*)
- each player in the game is aware of the sequence, strategies, and payoffs throughout gameplay (*complete information*)

Definition: A game has a finite set of "players" N, and each player $i \in N$ has a non-empty actions/strategies set $\{A_i\}_{i \in N}$. For each $i \in N$ there is a preference relation $\succcurlyeq_i$ on the set $A = \times_{j \in N} A_j$. Hence, a game can be stated as $\langle N, \{A_i\}_{i \in N}, \{\succcurlyeq_i\}_{i \in N} \rangle$

Notes

- In normal form games the terms action and strategy do not differ. This is no longer true in extensive form games that we will analyze later.

  A strategy $s_i \in \{S_i\}_{i \in N}$ of a player $i \in N$ is a function that assigns an action to each point (node or information set) where the specific player has to make a choice.

- We define the preference relation of a player i not on the set of his actions alone but rather on the set $A = \times_{j \in N} A_j$, the product of all players' action sets. The reason is because the choices of others affect the payoffs of player i.
- If A_i is finite for all the players $i \in N$ then the game is finite.

A utility function $u_i: A \rightarrow \mathcal{R}$, such that $\mathrm{u_i(a)} \geq \mathrm{u_i(b)}$ iff $a \succcurlyeq_i b$, may represent preferences. In such a case the game can be stated as $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$

- If a strategic form game consists of $N \leq 3$ players then it can be represented using payoff tables (matrices)

Example 1: Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the principal charge, but they have enough to convict both on a lesser charge. Simultaneously, the prosecutors offer each prisoner a bargain. Each prisoner is given the opportunity either to betray the other by testifying that the other committed the crime, or to cooperate with the other by remaining silent. The possible outcomes are:

- If prisoners 1 and 2 each betray the other, each of them serves six years in prison
- If prisoner 1 betrays prisoner 2 but prisoner2 remains silent, prisoner 1 will be set free and prisoner 2 will serve nine years in prison (and vice versa)
- If prisoners 1 and 2 both remain silent, they will both serve only one year in prison (on the lesser charge).

... Example 1: The described typical "prisoner's dilemma" game can be represented in the table below

		Player 2	
		Lie	Confess
Player 1	Lie	-1, -1	-9, 0
	Confess	0, -9	-6, -6

We can define all parts of the game, that is:

- $\mathcal{N} = \{1, 2\}$
- $A_1 = A_2 = \{C, L\}$
- $u_1(C, L) = u_2(L, C) = 0$
- $u_1(L, C) = u_2(C, L) = -9$
- $u_1(C, C) = u_2(C, C) = -6$
- $u_1(L, L) = u_2(L, L) = -1$

Example 2: Consider a duopoly where firms produce an identical product (at zero per unit cost) and "simultaneously" choose quantities (*i.e.*, competition à la Cournot). Let the inverse market demand be $\boldsymbol{p = 1 - q_1 - q_2}$.

We can define all parts of the game, that is:

- $\boldsymbol{\mathcal{N} = \{1, 2\}}$
- $\boldsymbol{A_1 = A_2 = [0, \infty)}$
- $\boldsymbol{\Pi_i(q_i, q_{-i}) = (1 - q_i - q_{-i})q_i}$

Dominant and Dominated Strategies

It is possible that

- it is always to the best interest of a player to choose the same action/strategy → ***strictly dominant action***
- it is never to the best interest of a player to choose a specific action/strategy → ***strictly dominated action***

Definitions:

1) A strategy $\hat{s}_i$ for player i is strictly dominant if $u_i(\hat{s}_i, s_{-i}) > u_i(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}$ and for all $s_i \neq \hat{s}_i \in S_i$
2) A strategy $\bar{s}_i$ for player i is strictly dominated if there exists some $s_i^* \in S_i$ such that $u_i(\bar{s}_i , s_{-i}) < u_i(s_i^*, s_{-i})$ for all $s_{-i} \in S_{-i}$

Dominant and Dominated Strategies

* It is intuitively clear that the outcome (*i.e.*, solution) of a situation that involves strategic interactions can NEVER include choices (i.e., actions/strategies) of the agents involved that are strictly dominated → all strictly dominated strategies can be eliminated

** However, a solution may very well include **weakly dominated strategies** (what's this?)

Dominant and Dominated Strategies

Note: If a game has a strictly dominant strategy for every player, the collection of these strategies is a (logical) solution of the game. For example, in the prisoner's dilemma game we get

- For player 1

$$\boldsymbol{u_1(C, L) = 0 > u_1(L, L) = -1}$$

$$\boldsymbol{u_1(C, C) = -6 > u_1(L, C) = -9}$$

→ player 1 has a strictly dominant strategy to confess (no matter if her partner confesses or lies!).

- For player 2

$$\boldsymbol{u_2(L, C) = 0 > u_2(L, L) = -1}$$

$$\boldsymbol{u_2(C, C) = -6 > u_2(L, C) = -9}$$

→ player 2 has a strictly dominant strategy to confess (no matter if his partner confesses or lies!).

**Hence, a logical solution/outcome of the prisoner's dilemma game is that BOTH players choose to confess.

- Is this concept enough to analyze all games? NO. There can be infinitely many games without a strictly dominant strategy for all players.

Iterated Elimination of Strictly Dominated Strategies

Rational players DO NOT play strictly dominated strategies. Therefore, in a game we can eliminate all strictly dominated strategies.

- be careful: once strictly dominated strategies have been eliminated one should check if, on the remaining non-eliminated strategies space, there are "revealed" strictly dominated strategies.

Example

Use the iterated elimination of strictly dominated strategies on the following game

		Player 2		
		Left	Middle	Right
Player 1	Up	1, 0	1, 2	0, 1
	Down	0, 3	0, 1	2, 0

Two caveats: The process of iterated elimination of s.d.s.

1. additional assumptions about players' rationality should be imposed.
2. need not lead to a "solution" of the game

Nash Equilibrium

The idea behind the Nash Equilibrium (NE) concept is simple: given the choices of all other players (no matter if these choices are "best," "rational," etc.) should I change my choice? No, I shouldn't IF I have chosen the action that yields the best for me (BEST RESPONSE) for the given choices of others!

Definition: A best response function (or correspondence) is defined as

$$\boldsymbol{B_i(a_{-i}) = \{a_i \in A_i | (a_i, a_{-i}) \succcurlyeq_i (a_i', a_{-i}), \forall a_i' \in A_i\}}$$

Definition: A Nash Equilibrium of a strategic form game is an action profile $\boldsymbol{a^* \in A = \times_{i \in \mathcal{N}} A_i}$ such that $\boldsymbol{(a_i^*, a_{-i}^*) \succcurlyeq_i (a_i, a_{-i}^*)}$ for all $\boldsymbol{a_i \in A_i}$ and for all $\boldsymbol{i \in \mathcal{N}}$.
Alternatively, A Nash Equilibrium is a profile $\boldsymbol{a^* \in A}$ such that $\boldsymbol{a_i^* \in B_i(a_{-i}) \, \forall i \in \mathcal{N}}$

Nash Equilibrium

How do we find the NE of a game? We must check every action profile if a player wants to deviate from that state for fixed choices others (i.e., unilateral deviation). If, for a given profile, there is at least one player that wants to change her action the given profile CANNOT be a Nash Equilibrium. If, for a given profile, no player wants to change his action then the given profile is a Nash Equilibrium.

Alternatively, we use the Best Response Functions and look for a profile that satisfies the alternative definition of NE.

Example 1': Consider again the typical "prisoner's dilemma" game as in the table below

		Player 2	
		Lie	Confess
Player 1	Lie	-1, -1	-9, 0
	Confess	0, -9	-6, -6

Example 2': Consider again a duopoly where firms produce an identical product (at zero per unit cost) and "simultaneously" choose quantities (*i.e.*, competition à la Cournot). Let the inverse market demand be $\boldsymbol{p = 1 - q_1 - q_2}$.

Nash Equilibrium: None, one, or many?

In the examples we have previously discussed we identified a unique Nash Equilibrium in each one. Is it always the case?

Example 3

Battle of the Sexes

		Player 2	
		UFC	Opera
Player 1	UFC	2, 1	0, 0
	Opera	0, 0	1, 2

(an example with two Nash equilibria)

Example 4

Matching the Pennies

		Player 2	
		Heads	Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

(an example with no (?) Nash Equilibrium)

Nash Equilibrium: None, one, or many?

Implications of example 3:

- When there are multiple Nash equilbria there is a tiny little problem! We cannot be sure as to which Nash equilibrium will prevail (or, IF any of the Nash equilibria will be realized at the end!).
- We will discuss later the problem of coordination failure.

Implications of example 4:

- In many cases it is essential to identify if a game has a solution (i.e., Nash equilibrium) or not!
- However, we will see that the game of "matching the pennies" has actually a Nash Equilibrium.

Time to practice!

Consider a Cournot triopoly where firms have zero marginal and zero fixed costs and the inverse demand is described by

$$p = \begin{cases} A - q_1 - q_2 - q_3, & if \sum_{i=1}^{3} q_i \leq A \\ 0, & otherwise \end{cases}$$

Find all Nash Equilibria of this game.