GAME THEORY

Assignment 1

Instructions

You must answer all questions of this assignment. Each question carries equal weight. This is an individual project and accounts for 10% of your final grade on this course. The project is due on 30/03/2026 in class. Late assignments WON'T be accepted.

Exercise 1

Last helicopter ride from a remote research station

Two researchers at a remote field station learn that a helicopter is leaving because of an incoming storm. The pilot explains that he can take one researcher under normal flying conditions, together with that researcher's essential luggage, files, data drives, samples, and research equipment. He can also take both researchers, but only if they travel with minimal weight, which means leaving behind most of their luggage and, crucially, a substantial part of their research material (such as field notes, samples, storage devices, or experimental equipment). Each researcher must decide independently and simultaneously whether to board.

Each researcher ranks outcomes as follows:

- Best (payoff 3): leave alone. In that case, the researcher can evacuate safely and take along the essential products of their work.
- Second best (payoff 2): leave together. Both escape the storm, but each must abandon valuable research materials, reducing the benefit of departure.
- Worst (payoff 1): remain at the station waiting for another flight, facing danger, delay, and uncertainty.

a. Fully describe the game and represent the situation with the use of a payoff matrix.
b. Is there any strictly dominated strategy for player 1? Is there any strictly dominated strategy for player 2? Explain your answer.
c. What is (are) the Nash Equilibrium (Equilibria) in pure strategies? Explain your answer.
d. Is there a coordination failure in this game? Explain your answer.
e. Let each player assign a probability distribution over her actions. What is the Nash Equilibrium in Mixed Strategies? Explain your answer.
f. Given you answer in part (e), do you want to change your answer in part (b)?

Suggested Answer

a. Strategic-form representation

Let the two researchers be Player 1 and Player 2. Each player has two available actions:

$$S_i = \{B, N\},$$

where B means "board the helicopter" and N means "not board / remain at the station."

The payoff matrix is:

	Player 2: B	Player 2: N
Player 1: B	2,2	3,1
Player 1: N	1,3	1,1

Explanation:

- If both board, both escape but must leave research material behind, so each receives payoff 2.
- If only one boards, the one who boards leaves alone and receives payoff 3, while the other remains and receives payoff 1.
- If neither boards, both remain at the station and each receives payoff 1.

b. Strictly dominated strategies

For Player 1:

If Player 2 chooses B, Player 1 receives:

$$u_1(B, B) = 2 > 1 = u_1(N, B).$$

If Player 2 chooses N, Player 1 receives:

$$u_1(B, N) = 3 > 1 = u_1(N, N).$$

Therefore, B strictly dominates N for Player 1.

By symmetry, B also strictly dominates N for Player 2.

Thus, for both players, "not board" is strictly dominated by "board."

c. Pure-strategy Nash equilibrium

Since boarding strictly dominates not boarding for both players, the unique pure-strategy Nash equilibrium is:

$$(B, B).$$

At (B, B), neither player wants to unilteraly deviate. If Player 1 deviates to N, her payoff falls from 2 to 1. Similarly, if Player 2 deviates to N, her payoff falls from 2 to 1.

Therefore, (B, B) is the unique Nash equilibrium in pure strategies.

d. Coordination failure

There is no coordination failure in this game since there is a unique Nash Equilibrium and this equilibrium is Pareto Efficient. To prove that (B, B) is Pareto Efficient note that an outcome is Pareto efficient if there is no other feasible outcome that makes at least one player strictly better off without making the other player worse off.

At (B, B), payoffs are:

$$u_1(B,B) = 2, u_2(B,B) = 2.$$

Now compare (B,B) with every other possible outcome.
1. Compare $(\boldsymbol{B},\boldsymbol{B})$ with $(\boldsymbol{B},\boldsymbol{N})$.

$$(B,N) = (3,1).$$

Player 1 is better off:

$$3 > 2.$$

But Player 2 is worse off:

$$1 < 2.$$

Therefore, (B,N) is not a Pareto improvement over (B,B).

2. Compare $(\boldsymbol{B},\boldsymbol{B})$ with $(\boldsymbol{N},\boldsymbol{B})$.

$$(N,B) = (1,3).$$

Player 2 is better off:

$$3 > 2.$$

But Player 1 is worse off:

$$1 < 2.$$

Therefore, (N,B) is not a Pareto improvement over (B,B).

3. Compare $(\boldsymbol{B},\boldsymbol{B})$ with $(\boldsymbol{N},\boldsymbol{N})$

$$(N,N) = (1,1).$$

Both players are worse off:

$$1 < 2.$$

Therefore, $(N, N)$is not a Pareto improvement over (B, B).

Since no other outcome makes one player better off without making the other player worse off, (B,B) is Pareto efficient.

e. Mixed-strategy Nash equilibrium
Let Player 2 board with probability q. Player 1's expected payoff from boarding is:

$$EU_1(B) = 2q + 3(1-q) = 3 - q.$$

Player 1's expected payoff from not boarding is:

$$EU_1(N) = 1q + 1(1-q) = 1.$$

Since $3 - q > 1$ for every $q \in [0,1]$, Player 1 strictly prefers B to N regardless of Player 2's mixed strategy. Similarly, Player 2 strictly prefers B to N regardless of Player 1's mixed strategy. Therefore, the unique mixed-strategy Nash equilibrium is degenerate:

$$p_1(B) = 1, p_2(B) = 1.$$

There is no non-degenerate mixed-strategy equilibrium.

f. Should the answer to part b change?
No. The mixed-strategy analysis confirms the answer in part b. Since boarding is strictly better for each player regardless of what the other player does, not boarding remains a strictly dominated strategy.

Exercise 2
Consider a static game between two players as described by the payoff matrix below:

		Player 2		
		Left	**Center**	**Right**
Player 1	**Up**	**2, 2**	**0, 1**	**2, 0**
	Middle	**3, 0**	**1, 1**	**0, 1**
	Down	**0, 2**	**1, 0**	**3, 3**

a. What are the actions/strategies sets for the two players?
b. Is there any strictly dominated strategy for player 1? Is there any strictly dominated strategy for player 2? Explain your answer.
c. What is (are) the Nash Equilibrium (Equilibria) in pure strategies? Explain your answer.
d. Is there a coordination failure in this game? Explain your answer.
e. Let each player assign a probability distribution over her actions. What is the Nash Equilibrium in Mixed Strategies? Explain your answer.
f. Given you answer in part (e), do you want to change your answer in part (b)?

Suggested Answers
a. Strategy sets
Player 1's strategy set is:
$$S_1 = \{Up, Middle, Down\}.$$

Player 2's strategy set is:
$$S_2 = \{Left, Center, Right\}.$$

b. Strictly dominated strategies
For Player 1, the payoff vectors are:
$$Up = (2,0,2),$$
$$Middle = (3,1,0),$$
$$Down = (0,1,3).$$

No strategy gives Player 1 a strictly higher payoff than another strategy against every action of Player 2. Therefore, Player 1 has no strictly dominated pure strategy.
For Player 2, the payoff vectors are:
$$Left = (2,0,2),$$

$$Center = (1,1,0),$$
$$Right = (0,1,3).$$

Again, no strategy gives Player 2 a strictly higher payoff than another strategy against every action of Player 1. Therefore, Player 2 has no strictly dominated pure strategy. Thus, neither player has a strictly dominated pure strategy.

c. Pure-strategy Nash equilibria

Find each player's best responses.

For Player 1:

If Player 2 chooses Left:

$$u_1(Up, Left) = 2, u_1(Middle, Left) = 3, u_1(Down, Left) = 0.$$

So Player 1's best response is $Middle$.

If Player 2 chooses Center:

$$u_1(Up, Center) = 0, u_1(Middle, Center) = 1, u_1(Down, Center) = 1.$$

So Player 1's best responses are $Middle$and $Down$.

If Player 2 chooses Right:

$$u_1(Up, Right) = 2, u_1(Middle, Right) = 0, u_1(Down, Right) = 3.$$

So Player 1's best response is $Down$.

For Player 2:

If Player 1 chooses Up:

$$u_2(Up, Left) = 2, u_2(Up, Center) = 1, u_2(Up, Right) = 0.$$

So Player 2's best response is $Left$.

If Player 1 chooses Middle:

$$u_2(Middle, Left) = 0, u_2(Middle, Center) = 1, u_2(Middle, Right) = 1.$$

So Player 2's best responses are $Center$and $Right$.

If Player 1 chooses Down:

$$u_2(Down, Left) = 2, u_2(Down, Center) = 0, u_2(Down, Right) = 3.$$

So Player 2's best response is $Right$.

The mutual best responses are therefore:

$$(Middle, Center)$$

and

$$(Down, Right).$$

Thus, the pure-strategy Nash equilibria are:

$$(Middle, Center), (Down, Right).$$

The associated payoffs are:

$$(Middle, Center): (1,1),$$
$$(Down, Right): (3,3).$$

d. Coordination failure

Yes, there is a possible coordination failure.

The game has two pure-strategy Nash equilibria:

$$(Middle, Center)$$

and

$$(Down, Right).$$

But $(Down, Right)$ Pareto-dominates $(Middle, Center)$, since both players receive payoff 3 instead of payoff 1

Therefore, if players coordinate on $(Middle, Center)$, they end up in a Nash equilibrium, but in an inefficient one. This is a coordination failure because both players would be better off at the alternative equilibrium $(Down, Right)$.

e. Mixed-strategy Nash equilibria

Let Player 1 choose:

Up with probability x_U,
$Middle$ with probability x_M,
$Down$ with probability x_D.

Let Player 2 choose:

$Left$ with probability y_L,
$Center$ with probability y_C,
$Right$ with probability y_R.

There are two non-pure mixed-strategy Nash equilibria.

Mixed equilibrium 1: support $\{Up, Down\}$ and $\{Left, Right\}$

Suppose Player 1 mixes between Up and $Down$, and Player 2 mixes between $Left$ and $Right$.

Let Player 2 choose $Left$ with probability q and $Right$ with probability $1 - q$.

Player 1's expected payoff from Up is:

$$EU_1(Up) = 2q + 2(1 - q) = 2.$$

Player 1's expected payoff from $Down$ is:

$$EU_1(Down) = 0q + 3(1 - q) = 3(1 - q).$$

For Player 1 to mix between Up and $Down$, she must be indifferent:

$$2 = 3(1 - q).$$

Therefore:

$$q = \frac{1}{3}.$$

So Player 2 plays:

$$Left \text{ with probability } \frac{1}{3},$$

$$Right \text{ with probability } \frac{2}{3},$$
$$Center \text{ with probability } 0.$$

Now let Player 1 choose Up with probability p and $Down$ with probability $1-p$.
Player 2's expected payoff from $Left$ is:
$$EU_2(Left) = 2p + 2(1-p) = 2.$$

Player 2's expected payoff from $Right$ is:
$$EU_2(Right) = 0p + 3(1-p) = 3(1-p).$$

For Player 2 to mix between $Left$and $Right$, she must be indifferent:
$$2 = 3(1-p).$$

Therefore:
$$p = \frac{1}{3}.$$

Thus, the first non-pure mixed-strategy Nash equilibrium is:
$$\left(\left(\frac{1}{3}, 0, \frac{2}{3}\right), \left(\frac{1}{3}, 0, \frac{2}{3}\right)\right),$$

where the order is:
$$(Up, Middle, Down)$$

for Player 1 and
$$(Left, Center, Right)$$

for Player 2.
So:
$$x_U = \frac{1}{3}, x_M = 0, x_D = \frac{2}{3},$$
$$y_L = \frac{1}{3}, y_C = 0, y_R = \frac{2}{3}.$$

<u>Mixed equilibrium 2: full-support mixed equilibrium</u>
Now suppose both players use all three strategies with positive probability.
Player 1 must be indifferent among Up, $Middle$, and $Down$. Given Player 2's mixed strategy (y_L, y_C, y_R), Player 1's expected payoffs are:
$$EU_1(Up) = 2y_L + 2y_R,$$
$$EU_1(Middle) = 3y_L + y_C,$$
$$EU_1(Down) = y_C + 3y_R.$$

Solving the indifference conditions gives:
$$y_L = y_C = y_R = \frac{1}{3}.$$

So Player 2 plays each action with probability 1/3.
Now Player 2 must be indifferent among $Left$, $Center$, and $Right$. Given Player 1's mixed strategy (x_U, x_M, x_D), Player 2's expected payoffs are:

$$EU_2(Left) = 2x_U + 2x_D,$$
$$EU_2(Center) = x_U + x_M,$$
$$EU_2(Right) = x_M + 3x_D.$$

Solving the indifference conditions gives:

$$x_U = \frac{1}{3}, x_M = \frac{5}{9}, x_D = \frac{1}{9}.$$

Thus, the full-support mixed-strategy Nash equilibrium is:

$$\left(\left(\frac{1}{3}, \frac{5}{9}, \frac{1}{9}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right).$$

Therefore, the non-pure mixed-strategy Nash equilibria are:

$$\left(\left(\frac{1}{3}, 0, \frac{2}{3}\right), \left(\frac{1}{3}, 0, \frac{2}{3}\right)\right)$$

and

$$\left(\left(\frac{1}{3}, \frac{5}{9}, \frac{1}{9}\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right).$$

The pure equilibria may also be viewed as degenerate mixed equilibria.

f. Should the answer to part b change?
No. The answer to part b should not change.
The fact that some strategies are not used in one mixed equilibrium does not mean that they are strictly dominated. In particular, the full-support mixed equilibrium uses every pure strategy of both players with positive probability. A strictly dominated strategy cannot be used with positive probability in equilibrium.
Therefore, no pure strategy is strictly dominated.

<u>Exercise 3</u>

Voluntary Contribution to a Public Good
Consider a group of n individuals, indexed by $i = 1,2, \ldots, n$. Each individual must simultaneously decide whether to contribute to the provision of a public good. Contributing entails a private cost $c > 0$. The public good is provided if at least one individual contributes. If the public good is provided, every individual receives a benefit $b > 0$, regardless of whether they contributed or not. Assume that
$b > c > 0$. Thus, provision of the public good is beneficial, but contributing is personally costly.

Let m denote the number of contributors in the group. The payoff of player i is given as follows:

- if $m = 0$, no public good is provided and each player receives payoff 0;
- if $m \geq 1$, the public good is provided:
  - each contributor receives $b - c$,
  - each non-contributor receives b.

a. Describe this situation as a strategic-form game. Clearly specify the set of players, the strategy set of each player, the payoff function.
b. Does any player have a strictly dominant strategy?
c. Find all Nash Equilibria of this game in pure strategies.

Suggested Answers
Let there be n players:

$$N = \{1,2, \dots, n\}.$$

Each player simultaneously chooses whether to contribute to the public good. Let:

$$C = \text{contribute},$$
$$NC = \text{not contribute}.$$

Assume:

$$b > c > 0,$$

where b is the benefit from the public good and c is the private cost of contributing.

a. Strategic-form game
The set of players is:

$$N = \{1,2, \dots, n\}.$$

Each player's strategy set is:

$$S_i = \{C, NC\}.$$

Let

$$k(s)$$

denote the number of contributors under strategy profile s.
The payoff of player i is:

$$u_i(s) = \begin{cases} 0, & \text{if } k(s) = 0, \\ b - c, & \text{if } k(s) \geq 1 \text{ and } s_i = C, \\ b, & \text{if } k(s) \geq 1 \text{ and } s_i = NC. \end{cases}$$

The public good is provided if and only if at least one player contributes.

b. Strictly dominant strategies
No player has a strictly dominant strategy.
To see this, consider player i.
If no other player contributes, then player $i$compares:

$$u_i(C) = b - c$$

with

$$u_i(NC) = 0.$$

Since $b > c$, we have:

$$b - c > 0.$$

Therefore, if no one else contributes, player i strictly prefers to contribute.
But if at least one other player contributes, then the public good is already provided. In that case, player $i$compares:

$$u_i(C) = b - c$$

with

$$u_i(NC) = b.$$

Since $c > 0$, we have:

$$b > b - c.$$

Therefore, if at least one other player contributes, player i strictly prefers not to contribute.
Thus, the best action depends on what the other players do. Hence no player has a strictly dominant strategy.

c. Pure-strategy Nash equilibria

The pure-strategy Nash equilibria are exactly the strategy profiles in which exactly one player contributes. Formally, the set of pure-strategy Nash equilibria is:

$$\{s \in S: k(s) = 1\}.$$

There are therefore n pure-strategy Nash equilibria: one for each possible player being the unique contributor. To prove this, consider three cases.

1. suppose no player contributes. Then the public good is not provided, and every player receives payoff 0. But any single player could deviate to contributing and receive:

$$b - c > 0.$$

Therefore, a profile with no contributors is not a Nash equilibrium.

2. Second, suppose exactly one player contributes. The contributor receives:

$$b - c.$$

If she deviates to not contributing, then no one contributes, the public good is not provided, and she receives:

$$0.$$

Since

$$b - c > 0,$$

the contributor does not want to deviate.
Each non-contributor receives:

$$b.$$

If a non-contributor deviates to contributing, the public good is still provided, but she pays the cost c, so her payoff becomes:

$$b - c.$$

Since

$$b > b - c,$$

no non-contributor wants to deviate.
Therefore, any profile with exactly one contributor is a Nash equilibrium.

3. Third, suppose two or more players contribute. Then any contributor can deviate to not contributing. Since at least one other player still contributes, the public good remains provided. The deviating player's payoff rises from:

$$b - c$$

to

$$b.$$

Since

$$b > b - c,$$

a profile with two or more contributors is not a Nash equilibrium.

Therefore, the pure-strategy Nash equilibria are exactly the profiles with one and only one contributor.