Lecture 4

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Overview

- Mixed Strategies and N.E.
- Examples
- Existence of N.E.
- Examples

Overview

- **Mixed Strategies and N.E.**
- Examples
- Existence of N.E.
- Examples

Mixed Strategies

- The game of Matching Pennies

v_1 / v_2	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

- An example to a zero-sum game.
- There is no equilibrium where a player chooses one of the two strategies.
- ESDS, IESDS, N.E. give us nothing.
- What should we do?
- Suppose they both flip a fair coin to decide what they will play.

Mixed Strategies

- The game of Matching Pennies

v_1 / v_2	H	T
H	$1, -1$	$-1, 1$
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- An example to a zero-sum game.
- There is no equilibrium where a player chooses one of the two strategies.
- ESDS, IESDS, N.E. give us nothing.
- What should we do?

- Suppose they both flip a fair coin to decide what they will play.
- Would any of them have the incentive to deviate?

Mixed Strategies

- The strategy set S_i is composed by pure strategies $s_i : s_i \in S_i$.
- A mixed strategy σ_i is a probability distribution over S_i
- σ_i is a randomization over pure strategies
- For examples, in Matching Pennies $\sigma_1(H) = \frac{1}{3}$ and $\sigma_1(T) = \frac{2}{3}$ is a mixed strategy.
- More generally, we can have $\sigma_1(H) = p$ and $\sigma_1(T) = 1 - p$ where $p \in [0, 1]$
- There are infinitely many mixed strategies (even when we have finitely many pure strategies).

Mixed Strategies

- Instead of payoffs we have expected payoffs:
- While $v_i(s_i, s_{-i})$ is the payoff associated with pure strategies, $v_i(\sigma_i, \sigma_{-i})$ is the average of associated payoffs weighted with the probabilities of play.
- We can extend the definition of **best response** by using the expected payoff functions:

$$\begin{aligned} b_i(\sigma_{-i}) &= \sigma_i \\ & \quad s.t. \\ v_i(\sigma_i, \sigma_{-i}) &\geq v_i(\sigma'_i, \sigma_{-i}), \forall \sigma_{-i} \end{aligned}$$

Nash Equilibrium in Mixed Strategies

- A mixed strategy profile $\sigma^* = (\sigma^*_1, \sigma^*_2, ..., \sigma^*_n)$ is a **Nash Equilibrium** if

$$v_i(\sigma^*_i, \sigma^*_{-i}) \geq v_i(\sigma'_i, \sigma^*_{-i})$$

- In other words, $\sigma^*$is a **Nash Equilibrium** if all players use their best response functions,

$$\sigma^*_i = b_i(\sigma^*_{-i}), \forall i$$

- At Nash Equilibrium no player has an incentive to change behavior
- That is, no player has a profitable deviation.

Back to Matching Pennies

- What if each player flips a fair coin to decide what they will play?
- $\sigma_1^* (H) = \sigma_1^* (T) = \frac{1}{2}$ and $\sigma_2^* (H) = \sigma_2^* (T) = \frac{1}{2}$ is a N.E. in mixed strategies.
- Does any player have a profitable deviation? No!
- $v_1 (H, \sigma_2^*) = \frac{1}{2} * 1 + \frac{1}{2} * (-1) = 0$
- $v_1 (T, \sigma_2^*) = \frac{1}{2} * (-1) + \frac{1}{2} * 1 = 0$
- Let's see why the use of the fair coin makes sense...

Back to Matching Pennies

- Let's extend the payoff matrix allowing for mixed strategies.
- Let $\sigma_1 (H) = p, \sigma_1 (T) = 1 - p$ and $\sigma_2 (H) = q, \sigma_2 (T) = 1 - q$.
- We need to compute p and q so that the two players have no profitable deviation:

		q	$1-q$
	v_1 / v_2	H	T
p	H	$1, -1$	$-1, 1$
$1-p$	T	$-1, 1$	$1, -1$

- P1: "If P2 plays $\sigma_2 (H) = q, \sigma_2 (T) = 1 - q$ then, when I play H, I get $q * 1 + (1 - q) * (-1) = 2q - 1$"
- P1: "If P2 plays $\sigma_2 (H) = q, \sigma_2 (T) = 1 - q$ then, when I play T I get $q * (-1) + (1 - q) * 1 = 1 - 2q$"

Back to Matching Pennies

Similarly for P2:

- P2: "If P1 plays $\sigma_1 (H) = p, \sigma_1 (T) = 1 - p$ then, when I play H, I get $p * (-1) + (1 - p) * 1 = 1 - 2p$"
- P2: "If P1 plays $\sigma_1 (H) = p, \sigma_1 (T) = 1 - p$ then, when I play T I get $p * 1 + (1 - p) * (-1) = 2p - 1$"
- Given the payoffs, let's find the best responses:

- For P1 we need to compare $2q - 1$ (payoff from H) and $1 - 2q$ (payoff from T).

Back to Matching Pennies

Similarly for P2:

- P2: "If P1 plays $\sigma_1 (H) = p, \sigma_1 (T) = 1 - p$ then, when I play H, I get $p * (-1) + (1 - p) * 1 = 1 - 2p$"
- P2: "If P1 plays $\sigma_1 (H) = p, \sigma_1 (T) = 1 - p$ then, when I play T I get $p * 1 + (1 - p) * (-1) = 2p - 1$"
- Given the payoffs, let's find the best responses:

- For P1 we need to compare $2q - 1$ (payoff from H) and $1 - 2q$ (payoff from T).
- For P2 we need to compare $1 - 2p$ (payoff from H) and $2p - 1$ (payoff from T).

Back to Matching Pennies

Similarly for P2:

- P2: "If P1 plays $\sigma_1(H) = p, \sigma_1(T) = 1 - p$ then, when I play H, I get $p * (-1) + (1 - p) * 1 = 1 - 2p$"
- P2: "If P1 plays $\sigma_1(H) = p, \sigma_1(T) = 1 - p$ then, when I play T I get $p * 1 + (1 - p) * (-1) = 2p - 1$"
- Given the payoffs, let's find the best responses:

- For P1 we need to compare $2q - 1$ (payoff from H) and $1 - 2q$ (payoff from T).
- For P2 we need to compare $1 - 2p$ (payoff from H) and $2p - 1$ (payoff from T).
- When the payoffs are equal I have no incentive to deviate (best response).

Back to Matching Pennies

For P1:

- For $q > 1/2$ we have $2q - 1 > 1 - 2q$ which means I should play H $(p = 1)$
- For $q < 1/2$ we have $2q - 1 < 1 - 2q$ which means I should play T $(p = 0)$
- For $q = 1/2$ we have $2q - 1 = 1 - 2q$ which means I am indifferent $(p \in [0, 1])$

For P2:

- For $p < 1/2$ we have $1 - 2p > 2p - 1$ which means I should play H $(q = 1)$
- For $p > 1/2$ we have $1 - 2p < 2p - 1$ which means I should play T $(q = 0)$
- For $p = 1/2$ we have $1 - 2p = 2p - 1$ which means I am indifferent $(p \in [0, 1])$
- Graph...

Overview

- Mixed Strategies and N.E.
- **Examples**
- Existence of N.E.
- Examples

Battle of the sexes

		q	$1-q$
	v_1/v_2	O	F
p	O	$2,1$	$0,0$
$1-p$	F	$0,0$	$1,2$

- P1: "If P2 plays $\sigma_2(O) = q, \sigma_2(F) = 1-q$ then, when I play O, I get $q*2+(1-q)*0 = 2q$"
- P1: "If P2 plays $\sigma_2(O) = q, \sigma_2(F) = 1-q$ then, when I play F I get $q*0+(1-q)*1 = 1-q$"
- P2: "If P1 plays $\sigma_1(O) = p, \sigma_1(F) = 1-p$ then, when I play O, I get $p*(1)+(1-p)*0 = p$"
- P2: "If P1 plays $\sigma_1(O) = p, \sigma_1(F) = 1-p$ then, when I play F I get $p*0+(1-p)*(2) = 2(1-p)$"

Battle of the sexes

For P1:

- For $q > 1/3$ we have $2q > 1 - q$ which means I should play O ($p = 1$)
- For $q < 1/3$ we have $2q < 1 - q$ which means I should play F ($p = 0$)
- For $q = 1/3$ we have $2q = 1 - q$ which means I am indifferent ($p \in [0, 1]$)

For P2:

- For $p > 2/3$ we have $p > 2(1 - p)$ which means I should play O ($q = 1$)
- For $p < 2/3$ we have $p < 2(1 - p)$ which means I should play F ($q = 0$)
- For $p = 2/3$ we have $p = 2(1 - p)$ which means I am indifferent ($p \in [0, 1]$)
- How many N.E. do we have?
- Graph...

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Existence of the N.E.

- Theorem (Nash, 1951): Every game where each player has finitely many pure strategies has a Nash equilibrium (possibly in mixed strategies).

Steps of the proof:

1. Recall that each mixed strategy is a probability distribution over the set of pure strategies.
2. Expected payoff is continuous in mixed strategies (see graphs).
3. Best response correspondence is continuous in mixed strategies.
4. Best response correspondence has a fixed point where each player's strategy is a best response to the others' strategies.

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Do the professionals play Nash Equilibrium?

- Research by Ignacio Palacios-Huerta (Review of Economic Studies, 2003)
- DATA: 5 years' worth of penalty kicks from Italian, Spanish and English football leagues.
- Empirical scoring probabilities (the Kicker is the row player and the Goalkeeper is the column player):

v_1 / v_2	L	R
L	58%	95%
R	93%	70%

- (L, L) is less probable than (R, R) probably because Right is the natural side for many kickers.

Do the professionals play Nash Equilibrium?

v_1 / v_2	L	R
L	58%	95%
R	93%	70%

- Payoffs: Scoring is +1 for kicker, -1 for goalie; no goal is 0.
- If the kicker plays L and the goalkeeper plays L then the former gets $+1 * 0,58 + 0 * 0,42 = 0,58$ whereas the latter gets $-1 * 0,58 + 0 * 0,42 = -0,58$
- Filling in the same way the table we have:

v_1 / v_2	L	R
L	$0,58, -0,58$	$0,95, -0,95$
R	$0,93, -0,93$	$0,7, -0,7$

- Is there a N.E. in pure strategies? What about in mixed?

Do the professionals play Nash Equilibrium?

		q	$1-q$
	v_1 / v_2	L	R
p	L	$0,58, -0,58$	$0,95, -0,95$
$1-p$	R	$0,93, -0,93$	$0,7, -0,7$

- The kicker is indifferent between L and R when $0,58q + 0,95(1-q) = 0,93q + 0,7(1-q) \rightarrow q = 0.42$
- The keeper is indifferent between L and R when $-0.58p - 0.93(1-p) = -0.95p - 0.7(1-p) \rightarrow p = 0.38$
- In fact, the observed probabilities (for strategy Left) are very close to our prediction using Nash Equilibrium: $p_{\text{real}} = 40\%$ and $q_{\text{real}} = 42\%$.
- Table...
- Also see Walker and Wooders (2001) for Wimbledon.

Mixed strategies in Hawk-Dove Game

		q	$1-q$
	v_1 / v_2	Aggressive	Passive
p	Aggressive	$0,0$	$4,2$
$1-p$	Passive	$2,4$	$3,3$

- In addition to the two asymmetric equilibria (Aggr, Pass) and (Pass, Aggr) in pure strategies,
- there exists a symmetric mixed-strategy equilibrium where the players randomize between the two alternatives with probabilities $q = p = 1/3$.
- Games in Biology: If you think of the two players as two members of the same species, this mixed strategy is an evolutionary stable strategy (John Maynard Smith, 1920 – 2004).

Reporting a Crime (Martin Osborne's textbook, Section 4.8)

- Catherine Genovese is murdered in New York in March 1964.
- 38 people witnessed the murder.
- None of them called the police or went for help.
- "Indifference to one's neighbor is a conditioned reflex of life in New York as it is in other big cities" concludes a social psychologist.
- Can we have a game theoretical explanation of this phenomenon?

Reporting a Crime (Martin Osborne's textbook, Section 4.8)

- n players, witnessing a crime.
- Each of them independently decides whether to call or not.
- Each gets the value v if the crime is reported by someone.
- Each incurs the cost $c < v$ if he/she chooses to report.
- An unreported crime brings payoff 0 to all.
- There are n pure-strategy Nash equilibria: One player calls the police. The others do not.
- Can we find a symmetric mixed-strategy Nash equilibrium where each player calls with probability p?

Reporting a Crime (Martin Osborne's textbook, Section 4.8)

- To find the equilibrium probability of calling, put yourself into the shoes of one of the players.
- You know that each of the other $n-1$ players would call with probability p.
- What is the probability the crime is not reported by one of them? $(1-p)^{n-1}$
- What is your payoff from reporting the crime? $v-c$
- ...from not reporting (given that somebody else does)? $v\left[1-(1-p)^{n-1}\right]$
- I am indifferent between the two when $v-c=v\left[1-(1-p)^{n-1}\right]$
- Probability that no player calls is $(1-p)^n=\left(\frac{c}{v}\right)^{\frac{n}{n-1}}$ is increasing in n!
- The more people we have, the less likely it is that the crime is reported.

Example (p.113 in the book)

v_1 / v_2	C	R
M	0, 0	3, 5
D	4, 4	0, 3

Are there any dominated strategies?
Can you make a prediction using the IESDS?
Find the N.E. in pure strategies.
First the N.E. in mixed strategies.

Example (p.113 in the book)

		q	$1-q$
	v_1 / v_2	C	R
p	M	$0, 0$	$3, 5$
$1-p$	D	$4, 4$	$0, 3$

For P1:

- Let's compare the payoffs when playing M $(3(1-q))$ and when playing D $(4q)$:
- For $3(1-q) > 4q \rightarrow \frac{3}{7} > q$ I should play M $(p = 1)$
- For $3(1-q) < 4q \rightarrow \frac{3}{7} < q$ I should play D $(p = 0)$
- For $3(1-q) = 4q \rightarrow \frac{3}{7} = q$ I am indifferent $(p \in [0, 1])$

Example (p.113 in the book)

		q	$1-q$
	v_1 / v_2	C	R
p	M	$0, 0$	$3, 5$
$1-p$	D	$4, 4$	$0, 3$

For P2:

- Let's compare the payoffs when playing C $(4(1-p))$ and when playing R $(5p + 3(1-p))$:
- For $4(1-p) > 5p + 3(1-p) \rightarrow \frac{1}{6} > p$ I should play C $(q = 1)$
- For $4(1-p) < 5p + 3(1-p) \rightarrow \frac{1}{6} < p$ I should play R $(q = 0)$
- For $4(1-p) = 5p + 3(1-p) \rightarrow \frac{1}{6} = p$ I am indifferent $(q \in [0, 1])$
- Graph...

Game Theory: Historical Perspective

Nobel Prizes in Economics:

- 1994 - John Nash, Reinhard Selten (Dynamic Games), and John Harsanyi (Bayesian Games).
- 1996 - James Mirrlees and William Vickrey: Incentives.
- 2001 - George Akerlof (lemons), Michael Spence (signaling), Joseph Stiglitz (screening): Asymmetric info.
- 2005 - Robert Aumann (repeated games) and Thomas Schelling (multiple equilibria).
- 2007 - Leonid Hurwicz, Eric Maskin, Roger Myerson: mechanism design.
- 2014 - Jean Tirole: Market power and Regulation.
- 2016 - Oliver Hart and Bengt Holmström: Contracts.
- 2020 - Robert B. Wilson and Paul R. Milgrom: Auctions.

Summary

- Mixed Strategies and N.E.
- Examples: Matching pennies, Battle of the Sexes, an example from the book
- Existence of N.E.
- Real world examples

The end

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